Answers

MATH 7B - WEEK 9

Name: ______________________
Section Number: ______________________

1 Solve the following pure-time initial value problem $\frac{dN}{dt} = \sqrt{t+9}$ where $N(0) = 20$.

$\int dN = \sqrt{t+9}\, dt$

$N = \frac{2}{3}(t+9)^{3/2} + C$

$20 = \frac{2}{3}(9)^{3/2} + C$

$20 = 18 + C$

$C = 2$

$\boxed{N = \frac{2}{3}(t+9)^{3/2} + 2}$

2a Solve the autonomous differential equation $\frac{dy}{dx} = y + 1$.

$\int \frac{1}{y+1}\, dy = \int dx$

$\ln|y+1| = x + C$

$y + 1 = e^{x+c}$ $\quad y = Ae^x - 1$

$\boxed{y = Ce^x - 1}$

2b Solve the autonomous initial value problem $\frac{dy}{dx} = y^2 + y$ where $y(0) = 1$.

$\int \frac{1}{y(y+1)}\, dy = \int dx$

$1 = A(y+1) + B(y)$

$A = 1, B = -1$

$\int \frac{1}{y} - \frac{1}{y+1}\, dy = \int dx$

$\ln|y| - \ln|y+1| = x + C$

$\ln\left|\frac{y}{y+1}\right| = x + C$

$\frac{y}{y+1} = Ce^x$

$y = Ce^x(y+1)$

$y = yCe^x + Ce^x$

$y(1 - Ce^x) = Ce^x$

$y = \frac{Ce^x}{1 - Ce^x}$

$1 = \frac{C}{1-C}$

$1 - C = C$

$1 = 2C$

$C = \frac{1}{2}$

$\boxed{y = \frac{\frac{1}{2}e^x}{1 - \frac{1}{2}e^x}}$

3. Solve the separable differential equation $\frac{dy}{dx} = 2xe^{-y}$.

$\int e^y\, dy = \int 2x\, dx$

$e^y = x^2 + C$

$\boxed{y = \ln(x^2 + C)}$

4. Solve the following separable differential equations with their initial values.

(a) $y' = 6y^2x$ where $y(1) = \dfrac{1}{25}$

$$\int \frac{dy}{y^2} = \int 6x\,dx$$

$$-\frac{1}{y} = 3x^2 + c$$

$$-25 = 3 + c$$

$$c = -28$$

$$-\frac{1}{y} = 3x^2 - 28$$

$$\boxed{y = \frac{1}{28 - 3x^2}}$$

(b) $\dfrac{dy}{dt} = e^{y-t}\sec(y)(1+t^2)$ where $y(0) = 0$.

$$\int e^{-y}\cos(y)\,dy = \int e^{-t}(1+t^2)\,dt$$

I.B.P.

$$\frac{e^{-y}}{2}(\sin y - \cos y) = -e^{-t}(t^2+2t+3) + c$$

$$\frac{1}{2}(-1) = -(3) + c \Rightarrow c = \frac{5}{2}$$

$$\frac{e^{-y}}{2}(\sin y - \cos y) = -e^{-t}(t^2+2t+3) + 5/2$$

5. Suppose that an object has a temperature T and is brought into a room that is kept at a constant temperature T_a. Newton's law of cooling states that the rate of temperature change of the object is proportional to the difference between the temperature of the object and the surrounding medium.

(a) Denote the temperature at time t by $T(t)$.

$$\frac{dT}{dt} = k(T_a - T)$$

Derive the solution to the differential equation, assuming that at time $t = 0$, the temperature of the object is $T = T_0$.

$$\int \frac{dT}{T_a - T} = \int k\,dt$$

$$-\ln|T_a - T| = kt + c$$

$$\ln|T_a - T| = -kt + C$$

$$T_a - T = |e^{-kt+c}|$$

$$-T = Ce^{-kt} - T_a$$

$$T = T_a - Ce^{-kt}$$

$$T_0 = T_a - C \Rightarrow C = T_a - T_0$$

$$\boxed{T = T_a - (T_a - T_0)e^{-kt}}$$

$$T_a + (T_0 - T_a)e^{-kt}$$