Answers NAME: ____________________

1. Decide whether or not the following integrals converge or diverge. If they converge, find their values.

$x \neq 3$

(a) $\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}} dx$

$= \int_{-\infty}^{0} (3-x)^{-1/2} dx$

$\lim_{z \to -\infty} \int_z^0 (3-x)^{-1/2} = \lim_{z \to \infty} \left[-2\sqrt{3-x} \Big|_z^0 \right] = \lim_{z \to \infty} \left[-2\sqrt{3} + 2\sqrt{3-z} \right]$

$\to +\infty \Rightarrow \boxed{\text{Diverges}}$

(b) $\int_{-\infty}^{\infty} xe^{-x^2} dx = \int_{-\infty}^{0} xe^{-x^2} dx + \int_{0}^{\infty} xe^{-x^2} dx$

$\lim_{z \to -\infty} \int_z^0 xe^{-x^2} dx = \lim_{z \to -\infty} \left[-\frac{e^{-x^2}}{2} \Big|_z^0 \right] = \lim_{z \to -\infty} \left[-\frac{1}{2} + \frac{e^{-z^2}}{2} \right] = -\frac{1}{2}$

$\lim_{z \to \infty} \int_0^z xe^{-x^2} dx = \lim_{z \to \infty} \left[-\frac{e^{-x^2}}{2} \Big|_0^z \right] = \lim_{z \to \infty} \left[-\frac{e^{-z^2}}{2} + \frac{1}{2} \right] = \frac{1}{2}$

So $\int_{-\infty}^{\infty} xe^{-x^2} dx = -\frac{1}{2} + \frac{1}{2} = \boxed{0}$ $\boxed{\text{Converges}}$

(c) $\int_0^{\infty} \sin(x) dx$

$\lim_{z \to \infty} \int_0^z \sin x \, dx = \lim_{z \to \infty} \left(-\cos x \Big|_0^z \right) = \lim_{z \to \infty} \left(-\cos(z) + 1 \right)$

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$\Longrightarrow \boxed{\text{Diverges}}$

(d) $\int_{-\infty}^{\infty} \frac{1}{x^4} dx$ $x \neq 0$

$= \int_{-\infty}^{-1} \frac{dx}{x^4} + \int_{-1}^{0} \frac{dx}{x^4} + \int_{0}^{1} \frac{dx}{x^4} + \int_{1}^{\infty} \frac{dx}{x^4}$ If one diverges, then the whole thing does

$\lim_{z \to 0^+} \int_z^1 \frac{dx}{x^4} = \lim_{z \to 0^+} \left[\frac{-1}{3x^3} \Big|_z^1 \right] = \lim_{z \to 0^+} \left[-\frac{1}{3} + \frac{1}{3z^3} \right] \longrightarrow \infty$

$\boxed{\text{Diverges}}$

2. Use trig substitution to evaluate the following indefinite integrals.

(a) $\int \sin^5(x)\cos^9(x)dx$ $\quad u = \sin x \quad du = \cos x\, dx$

$= \int \sin^5(x)(\cos^2(x))^4 \cos(x)dx = \int \sin^5 x(1-\sin^2 x)^4 \cos x\, dx$

$= \int u^5(1-u^2)^4 du = \int u^{13} - 4u^{11} + 6u^9 - 4u^7 + u^5\, du$

$= \frac{u^{14}}{14} - \frac{u^{12}}{3} + \frac{3u^{10}}{5} - \frac{u^8}{2} + \frac{u^6}{6} + C$

$= \boxed{\frac{\sin^{14}x}{14} - \frac{\sin^{12}x}{3} + \frac{3\sin^{10}x}{5} - \frac{\sin^8 x}{2} + \frac{\sin^6 x}{6} + C}$

(b) $\int \cos^4(x)\sin^2(x)dx$

$= \int \cos^4 x(1-\cos^2 x)dx = \int(\cos^4 x - \cos^6 x)dx$

⋮ use half angle formula a few times $\left(\cos^2 x = \frac{1+\cos(2x)}{2}\right)$

$= \int \frac{\cos(2x)}{32} - \frac{\cos(4x)}{16} - \frac{\cos(6x)}{32} + \frac{1}{16}dx$

$= \boxed{\frac{\sin(2x)}{64} - \frac{\sin(4x)}{64} - \frac{\sin(6x)}{192} + \frac{1}{16}x + C}$

(c) $\int \tan^2(x)\sec^6(x)dx$

$= \int \tan^2 x(\tan^2 x+1)^2 \sec^2 x\, dx$ $\quad u = \tan x \quad du = \sec^2 x\, dx$

$= \int u^2(u^2+1)^2 du = \int u^6 + 2u^4 + u^2\, du = \frac{u^7}{7} + \frac{2u^5}{5} + \frac{u^3}{3} + C$

$= \boxed{\frac{\tan^7 x}{7} + \frac{2\tan^5 x}{5} + \frac{\tan^3 x}{3} + C}$

3. Evaluate the indefinite integral $\int \frac{2x-1}{x^2-2x+3}dx$, by eventually using the trig sub: $u = \sqrt{2}\tan(\theta)$.

complete the square: $\int \frac{2x-1}{x^2-2x+3}dx = \int \frac{2x-1}{(x-1)^2+2}$

$u = x-1 \Rightarrow u+1 = x \quad du = dx$

$= \int \frac{2u+2-1}{u^2+2}du = \int \frac{2u}{u^2+2}du + \int \frac{1}{u^2+2}du$

We will solve these integrals separately

work on next page ↓

first integral

$$v = u^2+2 \Longrightarrow dv = 2u\,du$$

$$\int \frac{2u}{u^2+2}\,du = \int \frac{dv}{v} = \ln|v| = \ln|u^2+2|$$

second integral

$$u = \sqrt{2}\tan\theta \Longrightarrow du = \sqrt{2}\sec^2\theta\,d\theta$$

$$\int \frac{du}{u^2+2} = \int \frac{\sqrt{2}\sec^2\theta\,d\theta}{2\tan^2\theta+2} = \frac{1}{\sqrt{2}}\int \frac{\sec^2\theta\,d\theta}{\tan^2\theta+1}$$

$$= \frac{1}{\sqrt{2}}\int \frac{\sec^2\theta\,d\theta}{\sec^2\theta} = \frac{1}{\sqrt{2}}\int d\theta = \frac{\theta}{\sqrt{2}}$$

$$= \frac{\arctan\left(\frac{u}{\sqrt{2}}\right)}{\sqrt{2}}$$

Put together

$$\int \frac{2x-1}{x^2-2x+3}\,dx = \int \frac{2u}{u^2+2}\,du + \int \frac{du}{u^2+2}$$

$$= \ln|u^2+2| + \frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= \boxed{\ln|(x-1)^2+2| + \frac{1}{\sqrt{2}}\arctan\left(\frac{x-1}{\sqrt{2}}\right) + C}$$