Math 31 - Week 10 (Final Study Guide)

Raymond Matson

Name: ______________________________
Student ID: ______________________________

Please note that I am not taking part in creating the final for this class. I do not guarantee that any of these questions will or will not appear on your final and I'm creating this study guide without any knowledge of what the final will look like.

If you are having trouble on some problems, I recommend working together with other students to find solutions. Please feel free to send me an email at raymond.matson@email.ucr.edu if you have any questions or concerns about this study guide. Good luck on your finals everyone!

1. Give an example of a linear equation and a nonlinear equation.

2. What are the three elementary operations allowed in the process of row reduction?

3. Use Gaussian elimination to solve the following systems of linear equations.

   (a) $$\left\{\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}\right\}$$

   (b) $$\left\{\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned}\right\}$$

   (c) $$\left\{\begin{aligned} 9x_1 - 3x_2 - 1x_3 &= 0 \\ 5x_1 - 7x_2 - 9x_3 &= 0 \\ 6x_1 - 6x_2 - 6x_3 &= 0 \end{aligned}\right\}$$

4. Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

   (a) $$\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$$

   (b) $$\begin{bmatrix} 2 & -3 & h \\ -6 & 9 & 5 \end{bmatrix}$$

5. What is the difference between *echelon form* and *reduced row echelon form*?

6. A ____________ in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

7. Consider the following system of linear equations.

$$\begin{Bmatrix} x_1 - 5x_3 = 1 \\ x_2 + x_3 = 4 \\ 0 = 0 \end{Bmatrix}$$

   (a) Variables x_1 and x_2 are called ________________ variables.
   (b) The other variable, x_3, is called a _______________ variable.

8. Let A be an $k \times m$ matrix and B be a $n \times k$ matrix.

   (a) What is the size of matrix AB?

   (b) What is the size of matrix BA?

   (c) Using matrix A or matrix B, is there a matrix C we can multiply by to get a $l \times k$ matrix? If so, what is the size of matrix C?

   (d) Using matrix A or matrix B, is there a matrix C we can multiply by to get a $m \times l$ matrix? If so, what is the size of matrix C?

9. Determine if the following set of vectors are linearly independent or linearly dependent.

   (a) $\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\}$

   (b) $\{\begin{bmatrix}0\\1\\5\end{bmatrix}, \begin{bmatrix}1\\2\\8\end{bmatrix}, \begin{bmatrix}4\\-1\\0\end{bmatrix}\}$

   (c) $\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\}$

   (d) $\{\begin{bmatrix}8\\9\\-3\end{bmatrix}, \begin{bmatrix}-3\\2\\4\end{bmatrix}, \begin{bmatrix}1\\-2\\-1\end{bmatrix}\}$

10. True or False: Homogeneous equations $Ax = 0$ are always consistent.

11. True or False: The equation $Ax = b$ is inconsistent if the associated augmented matrix does not have a pivot in each row.

12. True or False: The set of vectors $\{v_1, v_2, v_3, v_4\}$ in $\mathbb{R}^3$ are always linearly dependent.

13. True or False: The set of vectors $\{v_1, v_2\}$ in $\mathbb{R}^3$ are always linearly independent.

14. True or False: If the columns of matrix A span $\mathbb{R}^m$, then $Ax = b$ is consistent for every vector $b \in \mathbb{R}^m$.

15. True or False: If p is a solution to $Ax = b$ and v is a solution to $Ax = 0$ then $p + v$ is a solution to $Ax = b$.

16. Any $m \times n$ matrix, A, has an associated linear transformation, $T(x)$, where $T:$ ____ $\longrightarrow$ ____.

17. Is the linear transformation $\begin{bmatrix} -2 & 6 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ one-to-one? Is it onto?

18. Is the linear transformation $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ one-to-one? Is it onto?

19. Is the linear transformation $\begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix}$ one-to-one? Is it onto?

20. Let T be a linear transformation such that $T(u) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $T(v) = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$. What is $T(3u-2v)$?

21. True or False: Composition of linear transformations are linear.

22. True or False: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ cannot be one-to-one if $n > m$.

23. True or False: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if $n < m$.

24. True or False: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ cannot be onto if $n < m$.

25. True or False: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if $n > m$.

26. Is the matrix $\begin{bmatrix} 1 & 2 & -4 \\ 5 & -5 & 1 \end{bmatrix}$ invertible? If it is find its inverse.

27. Is the matrix $\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ invertible? If it is find its inverse.

28. Is the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ invertible? If it is find its inverse.

29. Let $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -\frac{1}{6} & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ \frac{1}{3} & 0 & -\frac{1}{6} \end{bmatrix}$, and $b = \begin{bmatrix} 12 \\ -2 \\ 3 \end{bmatrix}$. Without using row reduction, why is there a solution to $Ax = b$? Find a solution to $Ax = b$ without using row reduction.

30. Suppose A, B, and C are invertible matrices with inverses A^{-1}, B^{-1}, and C^{-1} respectively.

    (a) What is the inverse of AB?

    (b) What is the inverse of ABC?

    (c) What is the inverse of $C^T ABA^T$?

31. List the definition of a *vector space*.

32. Determine whether or not the following sets are vector space.

    (a) $H = \{\begin{bmatrix} x \\ y \end{bmatrix}$ such that $x \geq 0$ and $y \geq 0\}$.

    (b) $H = \{\begin{bmatrix} x \\ y \end{bmatrix}$ such that $xy \geq 0\}$.

    (c) $H = \{\begin{bmatrix} x \\ 0 \end{bmatrix}$ such that x is any real number$\}$.

    (d) $H = \{\begin{bmatrix} x \\ y \end{bmatrix}$ such that $x^2 + y^2 \leq 1\}$.

    (e) The line $y = x$ in $\mathbb{R}^2$.

    (f) The set of upper triangular 3×3 matrices.

    (g) $GL_n(\mathbb{R}) = \{n \times n$ invertible matrices$\}$.

    (h) The set of all functions from $\mathbb{R}$ to $\mathbb{R}$.

    (i) $\mathbb{R}^n$ for any positive integer n.

    (j) The set of sequences with real values where addition is defined component wise.

    (k) The set of sequences with absolutely converging series.

    (l) $W \subset V$ where W is a subspace of vector space V.

33. Let V be a vector space. What are the requirements for W to be a *subspace* of V?

34. Determine whether or not the following are subspaces of $\mathbb{R}^3$.

    (a) $H = \{ \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$ such that $x = 2z\}$.

    (b) $H = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $3x + 4y = 5z\}$.

    (c) $H = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x^2 - y = z\}$.

    (d) $\mathbb{R}^2$

    (e) $S^2 = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x^2 + y^2 + z^2 = 1\}$.

    (f) $B^3 = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x^2 + y^2 + z^2 \leq 1\}$.

35. Determine whether or not the following are subspaces of the set of all functions from $\mathbb{R}$ to $\mathbb{R}$.

    (a) The set of polynomials of degree 1.

    (b) The set of polynomials of degree 7.

    (c) $\mathbb{P}^n$ = The set of polynomials of degree less than or equal to n for positive integer n.

    (d) $\mathbb{P}^4$ with irrational coefficients.

    (e) $\mathbb{P}^5$ with rational coefficients.

    (f) The set of real monic polynomials. (*Monic* means that the highest degree term has a coefficient of 1. For example, $x^2 + 2x - \pi$ is a monic polynomial.)

    (g) The set of all polynomials.

    (h) The set of continuous functions.

    (i) The set of differentiable functions.

Consider the bases $\mathcal{B} = \{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 5 \\ 1 \end{bmatrix} \}$ and $\mathcal{C} = \{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \}$.

36. Find vector u with coordinate vector $[u]_{\mathcal{B}} = \{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \}$.

37. Find vector v with coordinate vector $[v]_{\mathcal{C}} = \{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \}$.

38. Find the coordinate vector, $[u]_{\mathcal{B}}$, for $u = \{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 5 \\ 1 \end{bmatrix} \}$.

39. Find the coordinate vector, $[v]_{\mathcal{C}}$, for $v = \{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} \}$.

40. What two properties does a set of vectors need to be a basis for a vector space?

41. True or False: A basis for V is a spanning set for V that is as small as possible.

42. True or False: A basis for V is a linearly independent subset of V that is as large as possible.

43. True or False: If a finite subset S spans a vector space V, then some subset of S is a basis for V.

44. True or False: If $V = Span\{v_1, v_2, v_3\}$ then $\{v_1, v_2, v_3\}$ is a basis for V.

45. True or False: If a set $\{v_1, \cdots, v_p\}$ spans V then any subset of vectors in V with more than p vectors will be linearly dependent.

For the following problems, given a matrix A, provide a basis for the Nul(A) and Col(A). What is the rank of A? What are the dimensions of Nul(A) and Col(A)?

46. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

47. $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

48. $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

49. $A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 0 & 2 & -7 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

50. $A = \begin{bmatrix} 2 & -4 & 0 & 6 & 0 & -2 \\ 0 & 0 & 1 & -2 & 0 & 2 \\ 1 & -2 & 1 & 1 & 1 & -4 \\ 1 & -2 & 2 & -1 & 1 & -2 \\ 3 & -6 & 2 & 5 & 1 & -4 \end{bmatrix}$

51. Calculate the determinant of $A = \begin{bmatrix} -7 & 1 \\ -7 & -1 \end{bmatrix}$.

52. Calculate the determinant of $A = \begin{bmatrix} -6 & -5 & 4 \\ -6 & 3 & -9 \\ -4 & 1 & -5 \end{bmatrix}$.

53. Calculate the determinant of $A = \begin{bmatrix} -6 & 1 & -2 & 3 \\ -8 & -8 & -4 & 3 \\ -4 & -5 & -5 & -6 \\ -5 & 1 & -2 & 2 \end{bmatrix}$.

54. Use Cramer's rule to solve $Ax = b$ for $A = \begin{bmatrix} -2 & 2 & -4 & -1 & 7 \\ 0 & 3 & 7 & 2 & -9 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

55. Let $(0, 0, 0)$, $(5, 0, 0)$, $(0, 4, 0)$, $(0, 0, -3)$, $(5, 4, 0)$, $(5, 0, -3)$, $(0, 4, -3)$, and $(5, 4, -3)$ be the corner vertices of a parallelepiped in $\mathbb{R}^3$. Find the corresponding vectors that make up the edges of this parallelepiped and find its volume.

Determine whether or not you can diagonalize the following matrix A. If you can, provide matrices P and D such that $A = PDP^{-1}$.

56. $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{bmatrix}$.

57. $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$.

58. $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$.

59. $A = \begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

60. A is a 5×5 matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is A diagonalizable? Why or why not?

Finally, state the invertible matrix theorem with as many equivalent statements as possible.

Additionally, try writing just as many equivalent statements for non-invertible matrices.