Math 31 - Final Study Guide Solutions

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Please note that I am not taking part in creating the final for this class. I do not guarantee that any of these questions will or will not appear on your final and I'm creating this study guide without any knowledge of what the final will look like.

If you are having trouble on some problems, I recommend working together with other students to find solutions. Please feel free to send me an email at raymond.matson@email.ucr.edu if you have any questions or concerns about this linear algebra study guide. Good luck on your finals everyone!

1. Give an example of a linear equation and a nonlinear equation.
   Linear: $x_1 + 2x_2 = 0$; Nonlinear: $x_1^2 + x_1x_2 = 0$

2. What are the three elementary operations allowed in the process of row reduction?
   Scaling rows by nonzero constants, adding rows together, and swapping rows.

3. Use Gaussian elimination to solve the following systems of linear equations.

   (a) $\begin{Bmatrix} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 5x_1 - 5x_3 = 10 \end{Bmatrix}$ $\begin{array}{l} x_1 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{array}$

   (b) $\begin{Bmatrix} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 4x_1 - 8x_2 + 12x_3 = 1 \end{Bmatrix}$ No solutions

   (c) $\begin{Bmatrix} 9x_1 - 3x_2 - 1x_3 = 0 \\ 5x_1 - 7x_2 - 9x_3 = 0 \\ 6x_1 - 6x_2 - 6x_3 = 0 \end{Bmatrix}$ $\begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array}$

4. Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

   (a) $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$ $h \neq 2$

   (b) $\begin{bmatrix} 2 & -3 & h \\ -6 & 9 & 5 \end{bmatrix}$ $h \neq -\frac{5}{3}$

5. What is the difference between *echelon form* and *reduced row echelon form*?
   Echelon form requires (1) all nonzero rows above any rows of all zeros, (2) each leading entry of a row to be in a column to the right of the leading entry of the row above it, and (3) all entries in a column below a leading entry to be zeros.
   Reduced row echelon form additionally requires (4) the leading entry in each nonzero row to be 1 and (5) each leading 1 is the only nonzero entry in its column.

6. A <u>pivot position</u> in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

7. Consider the following system of linear equations.

$$\left\{\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}\right\}$$

   (a) Variables x_1 and x_2 are called <u>basic</u> variables.
   (b) The other variable, x_3, is called a <u>free</u> variable.

8. Let A be an $k \times m$ matrix and B be a $n \times k$ matrix.
   (a) What is the size of matrix AB? Can't multiply
   (b) What is the size of matrix BA? $n \times m$
   (c) Using matrix A or matrix B, is there a matrix C we can multiply by to get a $l \times k$ matrix? If so, what is the size of matrix C? Yes, CB, where C is a $l \times n$ matrix.
   (d) Using matrix A or matrix B, is there a matrix C we can multiply by to get a $m \times l$ matrix? If so, what is the size of matrix C? No

9. Determine if the following set of vectors are linearly independent or linearly dependent.

   (a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ Linearly Dependent

   (b) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} \right\}$ Linearly Independent

   (c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ Linearly Dependent

   (d) $\left\{ \begin{bmatrix} 8 \\ 9 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$ Linearly Independent

10. True or False: Homogeneous equations $Ax = 0$ are always consistent. True

11. True or False: The equation $Ax = b$ is inconsistent if the associated augmented matrix does not have a pivot in each row. False

12. True or False: The set of vectors $\{v_1, v_2, v_3, v_4\}$ in $\mathbb{R}^3$ are always linearly dependent. True

13. True or False: The set of vectors $\{v_1, v_2\}$ in $\mathbb{R}^3$ are always linearly independent. False

14. True or False: If the columns of matrix A span $\mathbb{R}^m$, then $Ax = b$ is consistent for every vector $b \in \mathbb{R}^m$. True

15. True or False: If p is a solution to $Ax = b$ and v is a solution to $Ax = 0$ then $p + v$ is a solution to $Ax = b$. True

16. Any $m \times n$ matrix, A, has an associated linear transformation, $T(x)$, where $T : \underline{\mathbb{R}^n} \longrightarrow \underline{\mathbb{R}^m}$.

17. Is the linear transformation $\begin{bmatrix} -2 & 6 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ one-to-one? Is it onto? One-to-one and onto.

18. Is the linear transformation $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ one-to-one? Is it onto? Not one-to-one. Not onto.

19. Is the linear transformation $\begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix}$ one-to-one? Is it onto? Not one-to-one. Not onto.

20. Let T be a linear transformation such that $T(u) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $T(v) = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$. What is $T(3u-2v)$?

$$\begin{bmatrix} -3 \\ 4 \\ -4 \end{bmatrix}$$

21. True or False: Composition of linear transformations are linear. True

22. True or False: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ cannot be one-to-one if $n > m$. True

23. True or False: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if $n < m$. False

24. True or False: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if $n > m$. False

25. True or False: The following determinant function, $det : \mathbb{R}^{n^2} \to \mathbb{R}$, that computes the determinant of a $n \times n$ matrix, is a linear map. False

26. Is the matrix $\begin{bmatrix} 1 & 2 & -4 \\ 5 & -5 & 1 \end{bmatrix}$ invertible? If it is find its inverse. Not invertible.

27. Is the matrix $\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ invertible? If it is find its inverse. $\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$

28. Is the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ invertible? If it is find its inverse. $\begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$

29. Let $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -\frac{1}{6} & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ \frac{1}{3} & 0 & -\frac{1}{6} \end{bmatrix}$, and $b = \begin{bmatrix} 12 \\ -2 \\ 3 \end{bmatrix}$. Without using row reduction, why is there a solution to $Ax = b$? Find a solution to $Ax = b$ without using row reduction.
A solution can be found by $x = A^{-1}b = \begin{bmatrix} 36 \\ -2 \\ 54 \end{bmatrix}$

30. Suppose A, B, and C are invertible matrices with inverses A^{-1}, B^{-1}, and C^{-1} respectively.
    (a) What is the inverse of AB? $B^{-1}A^{-1}$
    (b) What is the inverse of ABC? $C^{-1}B^{-1}A^{-1}$
    (c) What is the inverse of C^TABA^T? $(A^{-1})^TB^{-1}A^{-1}(C^{-1})^T$

31. List the definition of a *vector space*.

32. Determine whether or not the following sets are vector space.

    (a) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ such that } x \geq 0 \text{ and } y \geq 0 \right\}$. Not a vector space

    (b) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ such that } xy \geq 0 \right\}$. Not a vector space

    (c) $H = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \text{ such that } x \text{ is any real number} \right\}$. Vector space

    (d) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ such that } x^2 + y^2 \leq 1 \right\}$. Not a vector space

    (e) The line $y = x$ in $\mathbb{R}^2$. Vector space

    (f) The set of upper triangular 3×3 matrices. Vector space

    (g) $GL_n(\mathbb{R}) = \{n \times n \text{ invertible matrices}\}$. Not a vector space

    (h) The set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Vector space

    (i) $\mathbb{R}^n$ for any positive integer n. Vector space

    (j) The set of sequences with real values where addition is defined component wise. Vector space

    (k) The set of sequences with absolutely converging series. Vector space

    (l) $W \subset V$ where W is a subspace of vector space V. Vector space

33. Let V be a vector space. What are the requirements for W to be a *subspace* of V?

34. Determine whether or not the following are subspaces of $\mathbb{R}^3$.

    (a) $H = \left\{ \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \text{ such that } x = 2z \right\}$. Subspace

    (b) $H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } 3x + 4y = 5z \right\}$. Subspace

    (c) $H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } x^2 - y = z \right\}$. Not a subspace of $\mathbb{R}^3$

    (d) $\mathbb{R}^2$. Not a subspace of $\mathbb{R}^3$

    (e) $S^2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } x^2 + y^2 + z^2 = 1 \right\}$. Not a subspace of $\mathbb{R}^3$

    (f) $B^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } x^2 + y^2 + z^2 \leq 1 \right\}$. Not a subspace of $\mathbb{R}^3$

35. Determine whether or not the following are subspaces of the set of all functions from $\mathbb{R}$ to $\mathbb{R}$.

    (a) The set of polynomials of degree 1. Not a subspace

    (b) The set of polynomials of degree 7. Not a subspace

    (c) $\mathbb{P}^n$ = The set of polynomials of degree less than or equal to n for positive integer n. Subspace

    (d) $\mathbb{P}^4$ with irrational coefficients. Not a subspace

    (e) $\mathbb{P}^5$ with rational coefficients. Not a subspace

    (f) The set of real monic polynomials. (*Monic* means that the highest degree term has a coefficient of 1. For example, $x^2 + 2x - \pi$ is a monic polynomial.) Not a subspace

    (g) The set of all polynomials. Subspace

    (h) The set of continuous functions. Subspace

    (i) The set of differentiable functions. Subspace

Consider the bases $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

36. Find vector u with coordinate vector $[u]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. $v = \begin{bmatrix} -4 \\ 1 \\ 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

37. Find vector v with coordinate vector $[v]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. $v = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$

38. Find the coordinate vector, $[u]_{\mathcal{B}}$, for $u = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 5 \\ 1 \end{bmatrix}$. $[u]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

39. Find the coordinate vector, $[v]_{\mathcal{C}}$, for $v = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}$. $[v]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

40. What two properties does a set of vectors need to be a basis for a vector space?
    They need to be linearly independent and span the vector space

41. True or False: A basis for V is a spanning set for V that is as small as possible. True

42. True or False: A basis for V is a linearly independent subset of V that is as large as possible. True

43. True or False: If a finite subset S spans a vector space V, then some subset of S is a basis for V. True

44. True or False: If $V = Span\{v_1, v_2, v_3\}$ then $\{v_1, v_2, v_3\}$ is a basis for V. False

45. True or False: If a set $\{v_1, \cdots, v_p\}$ spans V then any subset of vectors in V with more than p vectors will be linearly dependent. True

For the following problems, given a matrix A, provide a basis for the Nul(A) and Col(A). What is the rank of A? What are the dimensions of Nul(A) and Col(A)?

46. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ which has no basis.
$\dim(\text{Nul}(A)) = 2$, $\text{Rank}(A) = \dim(\text{Col}(A)) = 0$

47. $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ which has no basis, $\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$.
$\dim(\text{Nul}(A)) = 0$, $\text{Rank}(A) = \dim(\text{Col}(A)) = 3$

48. $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ $\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$, $\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$.
$\dim(\text{Nul}(A)) = 1$, $\text{Rank}(A) = \dim(\text{Col}(A)) = 2$

49. $A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 0 & 2 & -7 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ $\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, $\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ 4 \\ -1 \end{bmatrix} \right\}$.
$\dim(\text{Nul}(A)) = 1$, $\text{Rank}(A) = \dim(\text{Col}(A)) = 3$

50. $A = \begin{bmatrix} 2 & -4 & 0 & 6 & 0 & -2 \\ 0 & 0 & 1 & -2 & 0 & 2 \\ 1 & -2 & 1 & 1 & 1 & -4 \\ 1 & -2 & 2 & -1 & 1 & -2 \\ 3 & -6 & 2 & 5 & 1 & -4 \end{bmatrix}$ $\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$,
$\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. $\dim(\text{Nul}(A)) = 3$, $\text{Rank}(A) = \dim(\text{Col}(A)) = 3$

51. Calculate the determinant of $A = \begin{bmatrix} -7 & 1 \\ -7 & -1 \end{bmatrix}$. 14

52. Calculate the determinant of $A = \begin{bmatrix} -6 & -5 & 4 \\ -6 & 3 & -9 \\ -4 & 1 & -5 \end{bmatrix}$. 30

53. Calculate the determinant of $A = \begin{bmatrix} -6 & 1 & -2 & 3 \\ -8 & -8 & -4 & 3 \\ -4 & -5 & -5 & -6 \\ -5 & 1 & -2 & 2 \end{bmatrix}$. 65

54. Use Cramer's rule to solve $Ax = b$ for $A = \begin{bmatrix} -2 & 2 & -4 & -1 & 7 \\ 0 & 3 & 7 & 2 & -9 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. $x = \begin{bmatrix} 4/3 \\ 4/3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

55. Let $(0,0,0)$, $(5,0,0)$, $(0,4,0)$, $(0,0,-3)$, $(5,4,0)$, $(5,0,-3)$, $(0,4,-3)$, and $(5,4,-3)$ be the corner vertices of a parallelepiped in $\mathbb{R}^3$. Find the corresponding vectors that make up the edges of this parallelepiped and find its volume.
Vectors: $\left\{ \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \right\}$. Volume $= |\det(A)| = 60$

Determine whether or not you can diagonalize the following matrix A. If you can, provide matrices P and D such that $A = PDP^{-1}$.

56. $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{bmatrix}$. $P = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

57. $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$. $P = \begin{bmatrix} -58 & 8 & 1 \\ -49 & 5 & 0 \\ 14 & 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

58. $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$. $P = \begin{bmatrix} 0 & 0 & -16 & -8 \\ 0 & 0 & 4 & 4 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

59. $A = \begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. $P = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

60. A is a 5×5 matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is A diagonalizable? Why or why not? Yes

Finally, state the invertible matrix theorem with as many equivalent statements as possible.

Additionally, try writing just as many equivalent statements for non-invertible matrices.