Upper bounds of the Cop Number - Notes

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Abstract

The game of Cops and Robbers can be thought of as a two-player game using some graph, $G(V, E)$, where one person plays some number of cops and the other person plays one robber. The person playing the cops chooses some vertices for the cops to start on and then the robber does the same. Starting with the cops, the players alternate turns moving from vertex to vertex by moving along an edge. The cops win if in some finite number of moves they can "capture" the robber by moving to the same vertex as the robber and the robber wins if the cops can never capture him. Mathematicians try to answer the canonical question that arises: for some connected graph, what is the smallest number of cops needed to ensure that the cops win? For an arbitrary graph of order n, Henri Meyniel conjectured that $c(G)$, the minimum number of cops needed, is $O(\sqrt{n})$. Some upper bounds have been found and after more recent works, this conjecture has been looking more and more promising. However, mathematicians are still a ways off from proving Meyniel's conjecture. In fact, for *any* fixed $\epsilon > 0$, we still don't even know if $c(G) = O(n^{1-\epsilon})$

In 1976, a mathematician from Princeton University, named Torrence D. Parsons, introduced the idea of pursuit-evasion games on graphs. He created the idea of being able to move around graphs from vertex to vertex, which was used to create the turn system and other foundations for the game. Although Parsons is typically given credit for creating these graph searching games, he did not introduce the field of cops and robbers; that title is given to Alain Quilliot.

Quilliot published his PhD thesis in 1978 using this idea of pursuit-evasion games on graphs. Using modern algebra and analysis, he uncovered several results about this game that showed converging results of winning strategies. Quilliot also showed some characteristics of graphs and metrics that helped particular strategies win by introducing a kind of ordering to the set of vertices to characterize these graphs. An interesting note is that in his paper he merely referred to his work as "Pursuit Games" and the name "Cops and Robbers" did not appear until the next publication of the topic, which was made independently in 1983 by Nowakowski and Winkler. Richard Nowakowski, Peter Winkler, and Alain Quilliot were the first ones to really set up this field in graph theory.

In 1987, Peter Frankl published a paper attempting to find the upper bound of the cop number. In this paper he discussed a bit about the lower bounds of cop numbers and was able to find the cop number for k-regular Cayley graphs as well as a few other specific graphs by analyzing their girths. He also mentioned that a colleague of his, Henri Meyniel, suggested a deep conjecture to the upper bound of the cop number. *Meyniel's Conjecture:* For a connected graph of order n,

$$c(G) = O(\sqrt{n})$$

Meyniel's conjecture is still just a conjecture, however, it looks very promising as mathematicians have proven this bound and much lower bounds for many specific graphs.

Definitions

1. Graphs will be written as $G(V, E)$
2. $V(G)$ or V. The *order of* G is $|V| = n$.
3. A *cycle* is a nontrivial path that has starts and ends at the same point.
4. A *loop* is an edge going from a vertex to itself. For our purposes we will ignore any loops as it would be redundant to the game.
5. A *tree* is a connected and cycleless graph.
6. The *girth* of a graph is the length of the shortest cycle.
7. A cut vertex is a vertex whose removal results in a disconnected graph.
8. The *neighborhood* of a vertex, v, denoted $N(v)$, is the set of all vertices connected to v through a single edge. We will use $N_i(v)$ to denote the set of all vertices with distance i away from v.
9. The *diameter* of a graph, D or $\mathrm{Diam}(G)$, is the supremum of all paths on G. Will talk about diameter as a path and as a length throughout.
10. $\sup_{v \in V} \deg(v) = \Delta$.
11. $\inf_{v \in V} \deg(v) = \delta$.
12. A path, $P \subset G$, is *isometric* if $\forall\ u, v \in V(P)$, $d_P(u, v) = d_G(u, v)$.
13. We say $f(n) = O(g(n))$ ("order $g(n)$") if
$$\limsup_{n\to\infty} \frac{|f(n)|}{g(n)} < \infty$$
14. Similarly, $f(n) = o(g(n))$ if
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$
15. Finally, $f(n) = \Theta(g(n))$ if
$$\limsup_{n\to\infty} \frac{|f(n)|}{g(n)} < \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{f(n)}{g(n)} > 0$$
16. In probability, we say an event holds *asymptotically almost surely* (a.a.s) if it holds with probability tending to 1 as $n \to \infty$.
17. A map $\phi : G \to G'$ is an *isomorphism* if $\phi(v_1v_2) = \phi(v_1)\phi(v_2)$ for every $v_1v_2 \in E(G)$.
18. A set $S \subset V$ is a called a *dominating set* if $v \in V \setminus S \Rightarrow v \in N(u)$ for some $u \in S$. The *dominating number* is $\gamma(G) = \inf |S|$.
19. $H \subset G$ is *k-guardable* if, after a finite number of rounds, k cops can move only in the vertices of H is such a way that if the robber moves into H at round t, then he will be captured at round $t + 1$.
20. The *cop number*, $c(G)$, is the minimum number of cops needed for the cops to win on graph G.

Preliminary Facts/Examples

Note: Unless stated otherwise, every graph is connected and contains no loops. We only care about connected graphs because a disconnected graph can simply be thought of as playing multiple games. For example, if a graph has two components then it's like we are playing two separate games at the same time as the cops will have to take into account both components.

Observation 1. *Let G be a tree. Then $c(G) = 1$.*

Whenever $c(G) = 1$, G is called a *cop-win* graph.

Theorem 1. *The Petersen graph is the smallest* 3*-cop-win graph*

By the "smallest" graph we mean that any graph with order less than 10 or any graph with order 10 not isomorphic to the Petersen graph will require no more than 2 cops. Due to the Petersen graph having many interesting properties, it allows for easy solutions to certain counterexamples.

Lemma 1. *An isometric path is* 1*-guardable.*

By definition of an isometric path, this should be fairly clear. The proof is a variation of Hall's matching theorem.

Outerplanar Graphs

Definition 1. *A planar graph G is a graph isomorphic to some G' that does not have any edges crossing each other.*

Definition 2. *An outerplanar graph G is a planar graph such that there exists an embedding $\iota : G \hookrightarrow S^1$ such that $\iota(e) \subset S^1$ or $\iota(e)$ is a chord $\forall e \in E$.*

Theorem 2. *If G is an outerplanar graph, then $c(G) \leq 2$.*

Proof. Suppose G is outerplanar and has no cut vertices. Label each vertex $v_0, \cdots, v_{n-1}$ in order $\Rightarrow v_i \backsim v_{i-1} \mod n$. Now label the endpoints of each chord $a_1, \cdots, a_k$.
Put c_1 at a_1 and c_2 at a_k. Define the *cop territory*, T_c, as the the vertices such that the robber cannot move to without crossing over c_1 or c_2 and $T_r := G \setminus T_c$ as the *robber territory*. The general strategy is for the cops to wait for each other to get the the next chord.
WLOG suppose c_1 is at a_i and c_2 is at a_j for $i < j$. Since a_i is connected to $\{a_{i_1}, a_{i_2}, \cdots\}$, if the robber is between a cop and the endpoint, we can increase the cop territory by moving across a chord. We will continue the next move of the cops based on cases.

1. Case 1: There are no chords from the T_r to a_i.
   Then our cop can safely move to a_{i+1}.

2. Case 2: There are chords from a_i into T_r.
   Let a_r be the furthest endpoint from a_i (the closest to a_j.

   (a) Case 2a: the robber is between a_i and a_r.
   Then c_2 moves to a_r.

   (b) Case 2b: the robber is between a_j and a_r.
   Then c_1 moves to a_r.

Each case has successfully increased T_c. Continue the algorithm until $T_c = V(G)$. Hence $c(G) \leq 2$.

Now suppose G is outerplanar and has cut vertices. Let $\boldsymbol{\lambda}$ be a partition of G such that each λ_i is an outerplanar subgraph with no cut vertices.
Pick $\lambda_i \in \boldsymbol{\lambda}$ to put c_1 and c_2 on.

1. Case 1: The robber is not on λ_i.
   Then retract G and move c_1 and c_2 until they are closer to the robber.

2. Case 2: The robber is on λ_i.
   Consider the embedding $\iota : \lambda_i \mapsto S^1$. Repeat the above algorithm until $T_c = V(G)$.

Hence $c(G) \leq 2$.

$\square$

Planar Graphs

Lemma 2. *Let P_1 and be an isometric path on G from v to w and let P_2 be an isometric path from v to w on $G \setminus (P_1 \setminus \{v, w\})$ for some $v, w \in V(G)$. Then $P_1 \bigcup P_2$ and the interior of $P_1 \bigcup P_2$ are 2-guardable.*

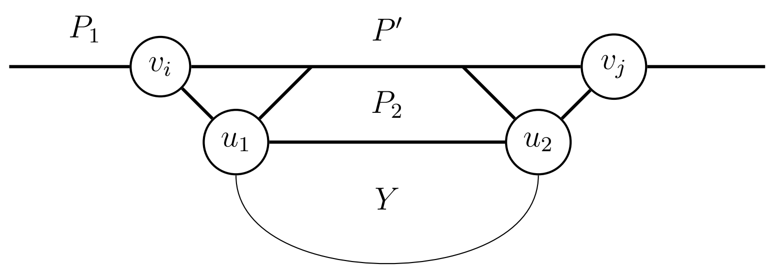
Theorem 3. *If G is an planar graph, then $c(G) \leq 3$.*

Proof. Once again let's consider cop territory and robber territory, T_c and T_r respectively. Let H be unguarded territory at any point of the game. There are 3 scenarios to consider.

1. Some cop is guarding an isometric path P in H. Any path from T_r to T_c is through P.
2. Two cops guard $P_1 \bigcup P_2$ where P_1 is isometric and P_2 is isometric in $G \setminus (P_1 \setminus \{v, w\})$ such that v and w are enpoints for P_1 and P_2.
3. A cop is on a cut vertex, v, of H such that any path from the T_r to T_c passes through v.

If at any point we are in case 3, the other cops, c_1 and c_2, move to c_3 and increase the cop number. Pick 2 vertices with maximum distance apart in G. Then there exists an isometric path P_1 connecting them. Guard P_2 with one cop, c_3, and move the other cops, c_1 and c_2, to Y.

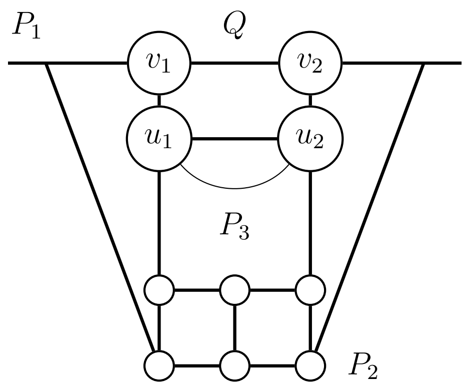
Choose two vertices of maximum distance from each other and let P_1 be an isometric path joining them. Begin by moving c_1 to guard P_1, giving us case 1 with $H = G$. Not the robber must remain on a component of $H \setminus P_1$. Let Y denote the component of H containing the robber.
If $!\exists\ v_i \in P_1$ adjacent to some vertex of Y, then v_i is a cut vertex of H. Move c_1 to v_i. Since c_1 is closer to v_i than the robber, the robber cannot escape from Y, giving us case 3. T_c then increases as described above.
Now suppose v_i and v_j, $i < j$, have neighbors in Y such that if v_r is another vertex of P_1 such that $P_1 \bigcap N[v_r] \neq \emptyset$, then $i < r < j$. Let $u_1 \in N[v_i] \bigcap Y$ and $u_2 \in N[v_j] \bigcap Y$. Now let P_2 be an isometric path in Y with u_1 and u_2 as end-vertices of P_2.



Move c_2 to guard P_2 while c_1 still guards the subpath of P' connecting v_i to v_j in P_1. The robber is either in the internal or external region bounded by the cycle by the cycle formed by $P' \bigcup P_2 \bigcup \{v_i u_1, v_j u_2\}$, giving us case 2.
Suppose we are in case 2. Let $X := P_1 \bigcup P_2$. WLOG assume the robber is in $\mathring{X}$ and $(\mathring{X})^C$ is guarded. This forces H to be the subgraph induced by G on $V(\mathring{X}) \setminus V(X)$. Notice that in this case $T_r \subset Y$. If $|V(X \bigcap N[Y])| = 1$, then this one vertex is a cut vertex of $\mathring{X}$. We can then move the free cop onto this vertex, giving us case 3.
Now suppose $|V(P_1)| = |V(P_2)| = 1$, v_1 and v_2 respectively, with neighbors in Y, so that we are not in the preceding subcase with a cut vertex. Now consider the subgraph, K, where

$$V(K) = V(Y) \bigcup \{v_1, v_2\} \text{ and } E(K) = E(Y) \bigcup \{v_i u \mid u \in V(Y) \text{ for } i = 1, 2\}$$

Let P be the shortest path from v_1 to v_2 in K. Since $v_1 \notin N[v_2]$ in $K \Rightarrow P \bigcap Y \neq \emptyset$. Move the free cop to guard P, giving us case 1 and therefore increasing T_c.
Now we will consider the case that one of the paths has two or more vertices with neighbors in Y. WLOG assume $v_1, v_2 \in V(P_1)$ have neighbors in Y and any other vertex of P_1 with neighbors in Y lie in the subpath $Q \subset P_1$ connecting v_1 and v_2. Similar to the preceding case with subgraph K, we will let P_3 be a shortest path from v_1 to v_2 containing a vertex of Y. Let v_1 be joined to $u_1 \in V(Y)$ and v_2 be joined to $u_2 \in V(Y)$ on P_3.



Move the free cop to guard P_3. If the robber is located in the region bounded by $V(P_3) \bigcup V(Q) \bigcup \{v_1u_1, v_2u_2\}$ then the cop guarding P_2 can move to guard Q, giving us case 2 with the cop guarding P_1 free to move around.
If the robber is not in the region bounded by $P_3 \bigcup Q \bigcup \{v_1u_1, v_2u_2\}$ then he's in the region bounded by P_2, P_3, and the subpaths of $P_1 \subset Q$. c_2 guards $P_3 \setminus \{v_1, v_2\}$ and P_3, as no path in Y is shorter. Define P' to be $P_3 \bigcup P_1 \setminus Q$, giving us case 2 with P_2 and P'. The cop guarding P_1 is now free to move around.
In each case, we have increased T_c after reaching the new case.

$\square$

This bound is sharp. A good exercise to see this is by showing that $c(G) \leq 3$ on a dodecahedron.

Random Graphs

In general, when we talk about a random graph, we can think of it fixing n vertices and with some probability, p, we add an edge between any two vertices. A random graph with order n and probability p is written as $G(n,p)$.

Theorem 4. *If* $p > \dfrac{(1-\epsilon)\log n}{n}$ *for* $\epsilon > 0$*, then* $\mathbb{P}(G$ *is connected*$) = 1$ *asymptotically almost surely. If* $p < \dfrac{(1-\epsilon)\log n}{n}$ *then* $\mathbb{P}(G$ *is connected*$) = 0$ *asymptotically almost surely.*

Theorem 5. *Let* $0 < p < 1$ *be fixed. For every real* $\epsilon > 0$*, a.a.s.*

$$(1-\epsilon)\log_{\frac{1}{1-p}} n \leq \gamma(G(n,p)) \leq (1+\epsilon)\log_{\frac{1}{1-p}} n$$

In particular, $\gamma(G(n,p)) = \Theta(\log n)$*.*

Theorem 6. *Let* $0 < p < 1$ *be fixed. For every real* $\epsilon > 0$*, a.a.s.*

$$(1-\epsilon)\log_{\frac{1}{1-p}} n \leq c(G(n,p)) \leq (1+\epsilon)\log_{\frac{1}{1-p}} n$$

In particular, $c(G(n,p)) = \Theta(\log n)$*.*

Proof. (of the lower bound)
Notice that if G is $(1,k)$-e.c., then $c(G) > k$. Let $0 < \epsilon < 1$ be fixed and let $k = \lfloor (1-\epsilon)\log_{\frac{1}{1-p}} n \rfloor$. Define $c = \log(\frac{1}{1-p})$ and $d = \frac{1-\epsilon}{c}$. Then $c, d > 0$ and $0 < cd < 1$. Then

$$\begin{aligned}\mathbb{P}(G \text{ is not } (1,k)\text{-e.c.}) &\leq n^{k+1}(1-p(1-p)^k)^{n-k-1} = e^{\log(n^{k+1}(1-p(1-p)^k)^{n-k-1})} \\ &\leq e^{(d\log n+1)\log(n)+(n-d\log n-1)\log(1-\frac{p}{n^{cd}})} = o(1)\end{aligned}$$

since $\log(1-\frac{p}{n^{cd}}) < 0$. Therefore a.a.s.

$$c(G(n,p)) \geq (1-\epsilon)\log_{\frac{1}{1-p}} n$$

The upper bound follows from the previous theorem and the clear observation of $c(G) \leq \gamma(G)$. $\square$

Observation 2. *If we choose the edges uniformly by letting* $p = 1/2$ *then* $c(G(n,p))$ *is around* $\log n$*, which is significantly smaller than Meyniel's bound, greatly supporting his conjecture.*

Peter Frankl 1987

Theorem 7.

$$c(G) = O(\frac{n \ln\ln n}{\ln n})$$

Proof. Start with the following equality and want in terms of D's and Δ's.

$$n = \sum_{i=0}^{D} |N_i| = 1 + \sum_{i=1}^{D} |N_i| \leq 1 + \sum_{i=1}^{D} \Delta(\Delta-1)^{i-1} = 1 + \sum_{i=0}^{D-1} \Delta(\Delta-1)^{i}$$

This is called the *Moore Bound* (Edward Moore 1925 - 2003)
Note: $|N_0| = 1$, $|N_1| \leq \Delta$, $|N_2| \leq \Delta(\Delta-1)$, $\cdots$, $|N_i| \leq \Delta(\Delta-1)^{i-1}$ (it's $\Delta - 1$: Don't double count the starting vertex)
Since just a geometric series

$$= 1 + \Delta\frac{(\Delta-1)^D - 1}{(\Delta-1)+1} = O(\Delta^D)$$

Claim: D and Δ can't both be less than $\frac{\ln n}{\ln\ln n}$. If you assume it then you'll get a graph with less than n vertices, even when using Moore's bound. (Plug into wolfram alpha and $O(\Delta^D)$ fails to be $\geq n$ when $n \geq 16$)

Let X be which ever one is greater than or equal to $\frac{\ln n}{\ln\ln n}$. If it's the diameter than X is an isometric path $\Rightarrow$ X is 1-guardable. If it's Δ then $v \bigcup N_1(v)$ is 1-guardable by placing a cop on v. This restricts the robber to a component of $G \setminus X$. Let $G' := G \setminus X$. Then

$$c(G) \leq c(G') + 1$$

Let $c(n)$ be the biggest cop number of any connected graph with n vertices. Then we can work inductively on above to get

$$c(G) \leq c(n) \leq c(n/2) + \frac{n/2}{\frac{\ln n}{\ln\ln n}}$$

Half of graph $(n/2)$ covered by 1-guardable areas. 1 cop per $\frac{\ln n}{ln\ln n}$, $n/2$ times.
Can think of as if we can guard α by 1 cop then

$$c(n) \leq c(n - \frac{n/2}{\alpha} * \alpha) + \frac{n/2}{\alpha} \Rightarrow c(G) = O(n\frac{\ln\ln n}{\ln n})$$

□

So basically, we consider the graph as a bunch of these 1-guardable areas and fill it up using 1 cop per area.

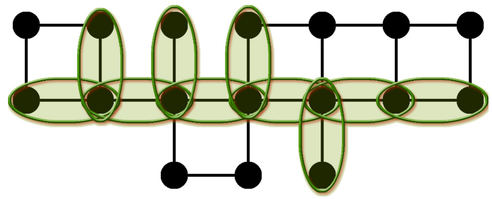
Ehsan Chiniforooshan 2008

$$c(G) = O(\frac{n}{\ln n})$$

Definition 3. *A subgraph $H \subset G$ is a Minimum Distance Caterpillar (MDC) iff*

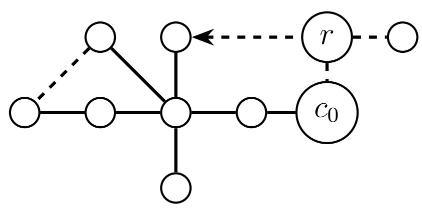
1. *H is a tree.*

2. *There exists an isometric path $P \subset H$ that is dominating.*



Lemma 3. *A MDC is 5-guardable*

Proof. (sketch)
Put 5 cops next to each other on the isometric path P, call these cops $c_{-2}, c_{-1}, c_0, c_1, c_2$. They shadow the robber throughout and it will be enough.



Contradiction version: Basically if 5 cops isn't enough, then your path isn't isometric.

□

Lemma 4. *There exists a MDC with order $\geq \ln n$. Equivalently, let $f(P) = |V(P) \bigcup N(P)|$. Then there exists isometric path $P \subset T$ such that $f(P) \geq \ln n$.*

Proof. (Proof by induction)
$n = 1, 2$ are trivially true. Suppose it's true for $k-1$ vertices. Suppose $|V(G)| = k$. Let r be an arbitrary vertex on G and consider a Breadth First-search tree rooted at r (basically means any root to leaf path is an isometric path) and call this subgraph T. Want to show there exists an isomorphic path of that is a MDC of length at least $\ln n$. Once again we will use induction.
Trivially true for $k = 1$ and $k = 2$. Assume true for $k \leq n-1$ and consider T such that $|V(T)| = n$. There exists $s \in N_T(r)$ such that there exists a subtree of $T \setminus \{r\}$, namely T_s, such that $|V(T_s)| \leq n-1$. Then by the inductive hypothesis $\exists$ P_s such that $|V(P_s)| \geq \ln(n-1)$. Now add r. Then

$$f(P) = f(P_s) + 1 + |N_T(r)| - 1 = \ln(\frac{n-1}{|N_T(r)|}) + |N_T(r)| = \ln(n-1) - \ln(|N_T(r)|) + |N_T(r)|$$

Notice that for any $0 < m \leq n$ for $n \geq 1.50245$, it's true that

$$m - \ln(m) \leq \ln n - \ln(n-1)$$

$$\Leftrightarrow \ln(n-1) + m - \ln(m) \leq \ln n$$

Thus

$$f(P) = \ln(n-1) - \ln(|N_T(r)|) + |N_T(r)| \geq \ln n$$

□

Combining the two lemmas we get

$$c(G) \leq c(G \setminus H) + 5$$

$$c(G) \leq c(n) \leq c(n - \ln n) + 5 \Rightarrow c(n) \leq c(n/2) + 5\frac{n/2}{\ln n}$$

So using the same method as before we get

$$c(n) \leq O(\frac{n}{\ln n}) + 5\frac{n/2}{\ln n} \Rightarrow c(G) \leq O(\frac{n}{\ln n})$$

Linyuan Lu and Xing Peng 2009

$$c(G) = O(\frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}})$$

Notation: $\Gamma_k(v) = \bigcup_{i=0}^k N_i(v)$

Theorem 8. *For any positive $k < n$, let $M_k = \min_{v \in V} |\Gamma_{2^{k-1}}(v)|$. Then*

$$c(G) \leq 8kn(\frac{\ln n}{M_k})^{1/k}$$

In particular, if the diameter of G is at most 2^{k-1}, we have $M_k = n$. Thus

$$c(G) \leq 8kn(\frac{\ln n}{n})^{1/k} = 8kn^{\frac{k-1}{k}} \ln^{1/k}(n)$$

Proof. The proof is disgusting and uses a lot of probability to prove. I will not be proving it. #SorryNotSorry.

□

Theorem 9. $c(G) = O(\frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}})$.

Proof. Basically we take the above theorem and let $f(x) = 8kx(\frac{\ln n}{n})^{1/k}$ and $k = \sqrt{\log_2 x}$. We want to show $c(G) \leq f(n)$ using induction.
When $n = 2 \Rightarrow k = 1$ (and $M_1 = 2$). By the above theorem $c(G) \leq 8 \ln 2$. (Just plugging in)
Suppose it's true for all G such that $|G| < n$.

1. *Case 1*: Diameter of G is $\leq 2^{k-1}$. Then again by above theorem,

$$c(G) \leq f(n) = 8n\sqrt{\log_2 n}(\frac{\ln n}{n})^{\frac{1}{\sqrt{\log_2 n}}}$$

2. *Case 2*: Diameter of G is $> 2^{k-1}$. Then there exists an isometric path P with length 2^{k-1} (can at least use a subpath of the diameter). This is 1-guardable. By induction hypothesis we have

$$c(G) \leq c(G \setminus P) + 1 \leq f(n - 2^{k-1}) + 1$$

Now consider $\dfrac{f(n) - f(n - 2^{\sqrt{\log_2 n} - 1})}{2^{\sqrt{\log_2 n} - 1}}$. By the mean value theorem, there exists m where $n - 2^{\sqrt{\log_2 n} - 1} \leq m \leq n$ such that

$$f'(m) = \frac{f(n) - f(n - 2^{\sqrt{\log_2 n} - 1})}{2^{\sqrt{\log_2 n} - 1}}$$

$$\Leftrightarrow f(n) - f(n - 2^{\sqrt{\log_2 n} - 1}) = f'(m) 2^{\sqrt{\log_2 n} - 1} > f'(n) 2^{\sqrt{\log_2 n} - 1} > 1$$

where the last inequality holds for $x \geq 3$.Hence

$$f(n) - f(n - 2^{\sqrt{\log_2 n} - 1}) > 1 \Leftrightarrow f(n) > f(n - 2^{\sqrt{\log_2 n} - 1}) + 1$$

Therefore,

$$c(G) \leq c(G \setminus P) + 1 \leq f(n - 2^{k-1}) + 1 \leq f(n)$$

Hence,

$$c(G) \leq f(n) = O(\frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}})$$

□

In the same paper, Lu and Peng proved the following result

Theorem 10. *If the diameter of a connected graph G is at most 2 or G is a bipartite graph with diameter at most 3, then $c(G) \leq 2\sqrt{n} - 1$.*

This result it tight up to a multiplicative factor. This result greatly suggests Meyniel's conjecture is correct as most graphs are of this type.

A Quick Comment About The Cop Number

Fix n vertices with no edges and consider different properties about the graph, such as the chromatic number, the dominating number, the connectivity, and the cop number. Now begin adding edges one by one to the graph and notice how all of those properties change. Every characteristic besides the cop number will monotonically change. However, the cop number begins at n, tends towards 1 as we get closer to a tree, then goes back up once large enough cycles appear, fluctuates wildly, and eventually goes back down to 1 when the graph becomes complete. This lack of monotonicity is one of the main reasons why finding a bound for the general cop number has proven so difficult.

Final Notes

Most of the research published in this field of graph theory is not on proving the upper bound all graphs in general. Many papers talk about specific types of graphs, such as the Peterson graph, series parallel graphs, and Cayley graphs. Some people like to alter the rules of the game a bit to see what happens. An example of this is catching a drunk robber on a graph, where the robber probabilistically moves around and we focus on how long it will take to capture the robber. Other variations include looking at the capture time when the cops are moving across multiple vertices or playing by going a particular direction down a hypercube, giving the robbers more of a chance to win. Mathematicians have also looked at the lower bound when then robber can move further per turn, when we don't constrain the robber by the edge, and even when we let the robber jump to any vertex on the graph (like a teleporting robber). Mathematicians have even looked at variations as modified as firefighters trying to put out a spreading fire or zombies vs. survivors, where the zombies are the cops and the survivors are the robbers. I have also done a bit of my own research in the subject by studying what happens when trying to capture multiple robbers simultaneously.

The entire idea of pursuit-evasion games on graphs brings up countless interesting questions that will most likely require very complex and creative solutions. This field of graph theory is a new and exciting one that is sure to spark interesting research and applications in the future. As far as Meyniel's conjecture is concerned, I personally believe that it is true and is only a matter of time before it is solved. It seems to me that the key to this solution will involve probabilistic methods and creative observations, yet only time will tell.