Abstract

Classical character theory is a way to condense important information of representations in order to classify groups up to isomorphism. Out of trying to characterize some particularly difficult cases, an even more condensed type of theory was discovered named supercharacter theories. We'll introduce supercharacter theory by talking about a particular motivating example called a pattern group. If there's time we will compare classical character tables to supercharacter tables.

Classical character theory for finite groups is a way to classify all finite groups using their representations. Groups are hard and sometimes it's easier to study how a portion of large vector space mimics the group structure. Since we only have to study the irreducible representations, we also only have to study irreducible characters. Unfortunately, there are some finite groups that are naturally more difficult to characterize (pun intended). As character theory for finite groups is mostly fleshed out, many mathematicians have ignored these difficult cases and have moved on to other areas of math to study. However, mathematicians that have tried to complete this are of research were forced to discover something else defined as a supercharacter theory. Although these supercharacter theories are very similar to character theories, sometimes supercharacters are the best we can do. Supercharacter theories give us a framework for "reducing" the character theory of groups while still maintaining the representation theoretic underpinnings.

It turns out that the groups that have been hindering the completion of the field of character theory are p-groups, and more specifically, groups called algebra groups. The goal of this talk will be to study a very specific algebra group called a pattern group and try to work out its characters/supercharacters.

Pattern Groups and Algebra Groups

Pattern groups are natrually occuring subgroups. So before we can talk about pattern groups, we need to understand its parent group.

Given a finite field with q elements, let

$$U_n(\mathbb{F}_q) := \{n \times n \text{ unipotent upper triangular matrices with entries in } \mathbb{F}_q\}$$

Definition 1. *$r \in R$, where R is some ring, is* ***unipotent*** *if $(r-1)^n = 0$ for some $n \in \mathbb{Z}_{>0}$.*

This is basically a fancy way of saying each diagonal entry for each element in this group is a 1. You should convince yourself that this is indeed a group over matrix multiplication, as $U_n(\mathbb{F}_q)$ contains the identity matrix, it's multiplicatively closed, and therefore contains inverses as the entries of these matrices are elements in a finite field.

Definition 2. *A subset $J \subseteq \{(i,j) \mid 1 \leq i < j \leq n\}$ is* ***closed*** *if $(i,j),(j,k) \in J \Rightarrow (i,k) \in J$.*

You should be thinking of J as a subset of indices above the diagonal of an $n \times n$ matrix. The closure of J allows matrix multiplication to be closed for the following subgroup.

For $t \in \mathbb{F}_q$, let $x_{ij}(t) \in U_n(\mathbb{F}_q)$ be the matrix whose (k,l)-th entry is determined the following way

$$(x_{ij}(t))_{kl} = \begin{cases} 1 & \text{if } k = l \\ t & \text{if } k = i \text{ and } l = j \\ 0 & \text{else} \end{cases}$$

Definition 3. *For any closed $J \subseteq \{(i,j) \mid 1 \leq i < j \leq n\}$,*

$$U_J := \quad < x_{ij}(t) \mid t \in \mathbb{F}_q, (i,j) \in J >$$

is called a ***pattern group***

For similar reasons why $U_n(\mathbb{F}_q)$ is a group, U_J is also a group. Also it's important to note that $U_J < U_n(\mathbb{F}_q) <$ some p-group (where technically $p = \frac{(q)(n)(n-1)}{2}$). Therefore, we really are looking at a very specific p-group case.

Examples:

1. $U_J = U_n(\mathbb{F}_q)$ is a pattern group.

2. The center of $U_n(\mathbb{F}_q)$ is U_J where $J = \{(1,n)\}$.

3. The upper and lower central series of $U_n(\mathbb{F}_q)$ has terms given by pattern groups where $J_k = \{(i,j) \mid j - i \geq k\}$.

4. The commutator subgroup $U'_J \subset U_n(\mathbb{F}_q)$ is equal to the Frattini subgroup $\Phi(U_n) \subset U_n(\mathbb{F}_q)$, which is the intersection of all maximal subgroups of $U_n(\mathbb{F}_q)$. In this case $U'_J = \Phi(U_n) = U_J$ where $J = \{(i,j) \mid 1 \leq i < i+1 < j \leq n\}$.
   In general, if J is closed then $U'_J = \Phi(U_J) = U_{J'}$ where $J' = \{(i,k) \mid (i,j),(j,k) \in J\}$.

Proposition 1. *For $q \geq 3$, a subgroup U of $U_n(\mathbb{F}_q)$ is a pattern group if and only if U is invariant under conjugation by diagonal matrices T in $GL_n(\mathbb{F}_q)$.*

Proof. ($\Rightarrow$) Suppose U is a pattern group and let $(u_{ij}) \in U$. Then

$$\operatorname{diag}(t_1, t_2, \cdots, t_n)(u_{ij}) \operatorname{diag}(t_1^{-1}, t_2^{-1}, \cdots, t_n^{-1}) = (t_i u_{ij} t_j^{-1})$$

Notice this still has one's on the diagonal, and any entries that were 0 in u_{ij} are still 0. Therefore, U is invariant under conjugation of T.

($\Leftarrow$) Now suppose U is invariant under conjugation of T. Let $u = (u_{ij}) \in U$ be nontrivial. It suffices to show for every $u_{ij} \neq 0$, the group $< x_{ij}(t) \mid t \in \mathbb{F}_q >$ is contained in U. Let i be minimal so that $u_{ij} \neq 0$ for some $i < j \leq n$. Let

$$h_i(t) = \operatorname{diag}(1, \cdots, 1, t, 1, \cdots, 1)$$

where t is in the ith position. Then $h_i(t)uh_i(t^{-1}) \in U$ has the effect of multiplying the ith row of u by t. If $t \neq 1$ then $u' := (h_i(t)uh_i(t^{-1}))u^{-1} \in U$. Notice that $u'_{jk} = 0$ when $j < k$ for $j \neq i$.
Now let j be minimal such that $u'_{ij} \neq 0$. j must exist by our choice of i. Then $h_j(t_2)uh_j(t_2^{-1}) \in U$ has the effect of multiplying the (i,j)th row entry by t_2^{-1}. If $t_2 \neq 1$ then $u'' := (h_j(t_2)u'h_j(t_2^{-1}))u'^{-1} \in U$. Notice that $u''_{kl} = 0$ unless $k = l$ and $l = j$.
Thus $u'' = x_{ij}(t)$ for some $t \in \mathbb{F}_q^\times$. Since U is invariant under T

$$< h_i(t_3)x_{ij}(t)h_i(t_3^{-1}) \mid t_3 \in \mathbb{F}_q^\times > \quad = \quad < x_{ij}(t_3 t) \mid t_3 \in \mathbb{F}_q^\times > \quad \subset \quad U$$

We can repeat inductively for every one-parameter subgroup in U to get our generators. Hence U is a pattern group.

$\square$

This proof fails when $q = 2$ as this proof requires that $|\mathbb{F}_q^\times| > 1$. In fact, if $q = 2$ then the diagonal condition is empty, making the proposition false. For example, if $s, t \in \mathbb{F}_2$ then

$$\begin{bmatrix} 1 & s & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \subseteq U_3(\mathbb{F}_2)$$

is a subgroup, however, it's not a pattern group since it doesn't contain $x_{12}(1)$. This proposition also implies that all characteristic subgroups, a subgroup that is mapped to itself by every automorphism of the parent group, of $U_n(\mathbb{F}_q)$ are pattern subgroups.
Another interesting note is that this proof doesn't depend on the field at all, other than the size constraint.

It turns out that $U_n(\mathbb{F}_q)$ is a specific type of algebra group.

Definition 4. *Let $\mathfrak{n}$ be a finite dimensional nilpotent $\mathbb{F}_q$ algebra. Then the corresponding* ***algebra group*** *is*

$$U_n = \{1 + X \mid X \in \mathfrak{n}\}$$

where multiplication is defined by

$$(1 + X)(1 + Y) = 1 + X + Y + XY.$$

A couple interesting notes:

- As an algebra group is a group that comes from a finite nilpotent algebra, every element in an algebra group is unipotent by construction.

- Algebra groups are a "type A" phenomenon (not every p-group is an algebra group).

- The center of an algebra group is an algebra group as $Z(U_n) = 1 + Z(\mathfrak{n})$.

- The classification of nilpotent algebras is incredibly difficult, which in turn makes the classification of algebra groups very difficult (although certain specific examples are known).

Theorem 1. *(Engel's Theorem)*
If a Lie algebra of matrices consists of nilpotent matrices, then the matrices can all be simultaneously brought to a strictly upper triangular form.

By this theorem we can view $\mathfrak{n}$ as a subalgebra of the set of $n \times n$ upper triangular matrices with zeroes on the diagonals. Therefore, every algebra group is isomorphic to a subgroup of $U_n(\mathbb{F}_q)$, hence why mathematicians who study supercharacters often study algebra groups and why we are talking about them in the first place.

Classical Group Representations and Characters

Definition 5. *A **matrix representation** of a group G is a homomorphism*

$$\rho : G \to GL_n(\mathbb{C})$$

For example, we can always have the trivial representation where $G \to [1]$. If $G = S_n$, we can consider the sign representation $G \to \{[1], [-1]\}$.

Definition 6. *A representation ρ of a group G on V is **irreducible** if V has no proper G-invariant subspace (i.e. $gv = v$ for $v \in V$ for all $g \in G$).*

Definition 7. *A **character**, denoted χ_ρ, is the complex-valued function such that $\chi_\rho(g) = Tr(\rho(g))$*

Irreducible representations and their corresponding characters give rise to a tabular display of these characters, which are called *character tables.* For example, the character tables for S_3 and $\mathbb{Z}/3\mathbb{Z}$ respectively are

	e	(12)	(13)	(23)	(123)	(132)
χ_{triv}	1	1	1	1	1	1
χ_{sgn}	1	-1	-1	-1	1	1
χ_{std}	2	0	0	0	-1	-1

and

	$\overline{0}$	$\overline{1}$	$\overline{2}$
χ_1	1	1	1
χ_2	1	$e^{\pi i/3}$	$e^{2\pi i/3}$
χ_3	1	$e^{2\pi i/3}$	$e^{\pi i/3}$

A couple interesting phenomena can be observed from this table.

1. The rows are orthogonal wrt the Hermitian product on characters ($< \chi, \chi' >= \frac{1}{|G|} \sum_g \overline{\chi(g)} \chi'(g)$).
2. $< \chi, \chi >= 1$ for irreducible character χ.
3. The conjugacy classes of G have the same character.

Definition 8. *Let G be an arbitrary finite group. A **supercharacter theory** for G is a partition κ of conjugacy classes, and a partition $\kappa^\vee$ of irreducible characters such that*

1. *The identity element has its own block in κ,*
2. *$|\kappa| = |\kappa^\vee|$,*
3. *For each block $K \in \kappa^\vee$, there exists a corresponding character χ^K which is a positive linear combination of the characters in K such that χ^K is constant on the blocks of κ.*

Alternatively, supercharacter theories can also be defined as the partitioning of G and its characters under the action of a group of automorphisms of G.

Roots

Let $R^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\}$ be a set of roots, which has a total order given by

$$\epsilon_r - \epsilon_s < \epsilon_i - \epsilon_j \quad \text{if } r > i \text{ or if } (r = i \text{ and } s > j)$$

Lemma 1. *The total order $\leq$ satisfies*

1. *If $\alpha < \beta$ and $\alpha + \beta \in R^+$, then $\alpha < \alpha + \beta < \beta$.*
2. *If $\alpha < \beta$, $\alpha + \beta \in R^+$, and $\alpha + \beta + \gamma \in R^+$, then either $\gamma < \alpha$ or $\beta < \gamma$.*

Proof. (a) If $\alpha < \beta$ and $\alpha + \beta \in R^+ \Rightarrow \alpha = \epsilon_j - \epsilon_k$ and $\beta = \epsilon_i - \epsilon_j$ for some $i < j < k \Rightarrow \alpha + \beta = \epsilon_i - \epsilon_k$. But since $i < j < k \Rightarrow \epsilon_j - \epsilon_k < \epsilon_i - \epsilon_k \Rightarrow \alpha < \alpha + \beta$. Similarly, $\epsilon_i - \epsilon_k < \epsilon_i - \epsilon_j \Rightarrow \alpha + \beta < \beta$.
(b) If $\alpha < \beta$ and $\alpha + \beta$, $\alpha + \beta + \gamma \in R^+ \Rightarrow \alpha = \epsilon_j - \epsilon_k$ and $\beta = \epsilon_i - \epsilon_j$. Thus as $\alpha + \beta = \epsilon_i - \epsilon_k \Rightarrow \alpha + \beta + \gamma$ has two choices: $\alpha + \beta + \gamma = \epsilon_a - \epsilon_k$ or $\alpha + \beta + \gamma = \epsilon_i - \epsilon_b$ for some ϵ_a, ϵ_b. But then $\gamma = \epsilon_a - \epsilon_i$ or $\gamma = \epsilon_k - \epsilon_b$. We will break this into 2 cases.

1) $\gamma = \epsilon_a - \epsilon_i$. Then by our definition $a < i \Rightarrow \epsilon_i - \epsilon_j < \epsilon_a - \epsilon_i \Rightarrow \beta < \gamma$.

2) $\gamma = \epsilon_k - \epsilon_b$. Then since $i < j < k \Rightarrow \epsilon_k - \epsilon_b < \epsilon_j - \epsilon_k \Rightarrow \gamma < \alpha$.

□

If we let $n = 4$ then the following Hasse diagram extended with transitivity shows the correspondence between the total ordering.

$$\begin{tikzcd}
& & \epsilon_1 - \epsilon_4 \arrow[dr] & & \\
& \epsilon_2 - \epsilon_4 \arrow[ur] \arrow[dr] & & \epsilon_1 - \epsilon_3 \arrow[dr] & \\
\epsilon_3 - \epsilon_4 \arrow[ur] \arrow[rr] & & \epsilon_2 - \epsilon_3 \arrow[uu] \arrow[ur] \arrow[rr] & & \epsilon_1 - \epsilon_2
\end{tikzcd}$$

If we think about each index of an upper triangular 5×5 matrix, we can also think of this ordering in the following way

$$\begin{tikzcd}
0 & a_{1,2} & a_{1,3} \arrow[l] & a_{1,4} \arrow[l] & a_{1,5} \arrow[l] \\
0 & 0 & a_{2,3} \arrow[urr] & a_{2,4} \arrow[l] & a_{2,4} \arrow[l] \\
0 & 0 & 0 & a_{3,4} \arrow[ur] & a_{3,5} \arrow[l] \\
0 & 0 & 0 & 0 & a_{4,5} \arrow[u] \\
0 & 0 & 0 & 0 & 0
\end{tikzcd}$$

More generally, if the roots are indexed by pairs (i,j), then this ordering for some n will be

$$(n-1,n) < (n-2,n) < (n-2,n-1) < \cdots < (1,n) < (1,n-1) < \cdots < (1,2)$$

Even though these roots have a total ordering there is a natural partial ordering that comes out of this.

Definition 9. *The* ***dominance order*** *of two roots, α and β, are as follows:*

$$\alpha \prec \beta \qquad \textit{if } \beta - \alpha \in R^+$$

As we're focusing on upper triangular matrices, the dominance order is more of what we care about as it creates a partial ordering on higher and higher diagonals. For example, let $n = 4$ and consider the following matrix:

$$\begin{bmatrix} 1 & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & 1 & a_{2,3} & a_{2,4} \\ 0 & 0 & 1 & a_{3,4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $\epsilon_1 - \epsilon_4$ is the greatest root and we can construct the following Hasse diagram.

$$\begin{tikzcd}
 & & \epsilon_1 - \epsilon_4 & & \\
 & \epsilon_2 - \epsilon_4 \arrow[ur] & & \epsilon_1 - \epsilon_3 \arrow[ul] & \\
\epsilon_3 - \epsilon_4 \arrow[ur] & & \epsilon_2 - \epsilon_3 \arrow[ul] \arrow[ur] & & \epsilon_1 - \epsilon_2 \arrow[ul]
\end{tikzcd}$$

We say a subset $J \subseteq R^+$ is *closed* if for every $\alpha, \beta \in J \Rightarrow \alpha + \beta \in J$. Let $\mathcal{G}(J)$ denote the subgraph of $\mathcal{G}(R^+)$ with vertices J and an edge from α and β if $\beta - \alpha \in J$. Following the above example where $n = 4$, $J = \{(1,2), (1,3), (1,4), (2,4), (3,4)\}$ is a closed subset. Then $\mathcal{G}(J) =$

$$\begin{tikzcd}
 & & \epsilon_1 - \epsilon_4 & & \\
 & \epsilon_2 - \epsilon_4 \arrow[ur] & & \epsilon_1 - \epsilon_3 \arrow[ul] & \\
\epsilon_3 - \epsilon_4 \arrow[ur] & & & & \epsilon_1 - \epsilon_2 \arrow[ul]
\end{tikzcd}$$

This gives us a way to talk about the main group we're focused on, pattern groups.

Roots of Pattern Groups

Why do we care about these types of roots? Well we can use these roots as a way to relate the closure of J to the closure of the pattern group.

For $\epsilon_i - \epsilon_j \in R^+$, denote

$$X_{ij} := X_{\epsilon_i - \epsilon_j} := n \times n \text{ matrix with 1 in the } (i,j)\text{th position and zeroes everywhere else.}$$

Then notice that the nilpotent $\mathbb{F}_q$-algebra, $\mathfrak{n}_J := \mathbb{F}_q\text{-span}\{X_\alpha \mid \alpha \in J\}$, has the following relations.

$$\begin{aligned} X_\alpha^2 &= 0 && \text{for } \alpha \in J, \\ X_\alpha X_\beta &= 0 && \text{for } \alpha, \beta \in J \text{ such that } \alpha < \beta \text{ OR } \alpha + \beta \notin R^+, \\ X_\beta X_\alpha &= X_{\alpha+\beta} && \text{for } \alpha, \beta, \alpha+\beta \in J, \alpha < \beta \text{ AND } \alpha + \beta \in R^+. \end{aligned}$$

In a sense, we will later want to think of the collection of these X_α's as our "vector space".
Let $J \subset R^+$ be closed, $\mathbb{F}_q$ be a finite field with q elements, and let

$$J^* = \left\{ \begin{aligned} \phi :& J \to \mathbb{F}_q \\ & \alpha \to \phi_\alpha \end{aligned} \right\}$$

Then $\mathfrak{n}_J = \{X_\phi \mid \phi \in J^*\}$ where $X_\phi = \sum_{\alpha \in J} \phi_\alpha X_\alpha$.

Digression: Suppose we want to find all finite dimensional irreducible representations of *some* algebra $\mathfrak{n}_J$. This can be done by looking at how our basis of $\mathfrak{n}$ acts on some k-dimensional vector space, W. If W is 0-dimensional then we're trivially done. If W is not 0-dimensional then we look how each basis element acts on this vector space, which should be linear. For example, if λ is a basis element, then $\lambda W = rW$ for some $r \in \mathbb{F}_q$. We continue until we can find all linearly independent vector spaces, telling us our irreducible representations.
Typically, we can talk about characters over some vector space V in the following way

$$ch(V) = \sum_r \dim V_r e^r \quad \Rightarrow \quad chV(r) = e^{-r} + \cdots + e^r$$

for $r \in \mathbb{F}_q$, which keeps track of the dimensions. It's important to note that due to linearity,

$$ch(V(\mu) \oplus V(\lambda)) = chV(\lambda)) + chV(\lambda) \quad \text{and} \quad ch(V(\mu) \otimes V(\lambda)) = chV(\mu)chV(\lambda)$$

In our case, the dual of $\mathfrak{n}_J$ is

$$\begin{aligned} \mathfrak{n}_J^* = \{\lambda :& \mathfrak{n}_J \to \mathbb{F}_q \mid \lambda \text{ is } \mathbb{F}_q\text{-linear}\}, \\ & X_\phi \mapsto \phi \end{aligned}$$

which has a basis, $\{\lambda_\alpha : \mathfrak{n}_j \to \mathbb{F}_q \mid \alpha \in J\}$, given by $\lambda_\alpha(X_\phi) = \phi_\alpha$.

To recap, we have shown generally what a pattern group is and then showed it again, using a more lie theoretic perspective. This can be seen be looking at the following correspondences.

$$\text{Roots} \longleftrightarrow J$$

$$\mathfrak{n}_J \text{ (The nilpotent } \mathbb{F}_q \text{ algebra)} \longleftrightarrow \mathfrak{n}_J^* \text{ (dual)} \longleftrightarrow 1 + \mathfrak{n}_j^* \text{ (algebra group)} \longleftrightarrow \ < x_\alpha(t) >$$

Supercharacter Formula for Pattern Groups

Recall:

- $\eta \in J^*$.
- $\lambda_\alpha : \mathfrak{n}_J \to \mathbb{F}_q$ is a basis for the dual of $\mathfrak{n}_J$.
- $\lambda_\eta = \sum_{\alpha \in J} \eta_\alpha \lambda_\alpha$.

We will define the following orbit as

$$O_\phi := \{U_J \times U_J - \text{orbit containing } X_\phi\}$$

the group acting on the nilpotent algebra $\mathfrak{n}$ by left and right multiplication and the co-orbit is

$$O^\eta := \{U_J \times U_J - \text{orbit containing } \lambda_\eta\} \quad \text{defined as} \quad x\lambda(X_\phi)y = \lambda(x^{-1}X_\phi y^{-1}),$$

the orbit of the group acting on $\mathfrak{n}$'s dual space.

Definition 10. *The* ***superclass*** *corresponding to $\phi \in J^*$ is $\{x_\rho \mid \rho \in O_\phi\}$, which is a union of conjugacy classes in U_J.*

So given an element in the dual, we consider all of these orbits of $\mathfrak{n}$ by left and right multiplication as the superclasses corresponding to that dual element.

Now given a superclass and fixed nontrivial homomorphism $\theta : \mathbb{F}_q^+ \to \mathbb{C}^\times$, we can create a supercharacter in the following way:

Observation 1. *A* ***supercharacter*** *of U_J is the map*

$$\begin{aligned}\chi^\eta(x_\phi) &:= \frac{q^{corank(\eta)}}{|O^\eta|} \sum_{\mu \in O^\eta} \theta(\lambda_\mu(X_\phi)) \\ &= \frac{q^{corank(\eta)}}{|O_\phi|} \sum_{\rho \in O_\phi} \overline{\theta(\lambda_\eta(X_\rho))}\end{aligned}$$

These maps are constant on the superclasses and are orthogonal under the usual inner product with the relation

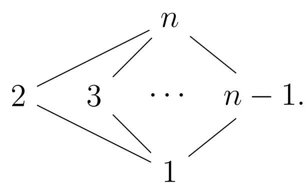
$$< \chi^\eta, \chi^\mu >= \delta_{\eta\mu} \frac{q^{2\ \text{corank}(\eta)}}{|O^\eta|}$$

Examples:

1. The Heisenberg group.
   Let H_n be the group of order q^{2n-3} represented as $n \times n$ matrices with ones on the diagonal, entries in $\mathbb{F}_q$, and non-zero entries allowed only in the top row or last column. Thus, this Heisenberg group is a pattern group U_J with

   $$J = \{(n-1, n), (n-2, n), \cdots, (1, n), (1, n-1), (1, n-2), \cdots, (1, 2)\}$$

   and the corresponding poset, $\mathcal{P} =$



The structure and character theory of H_n is well-known and so the following supercharacter can be verified.

$$\overline{\chi^{\eta}(x_{\phi})} = \begin{cases} \prod_{(i,j)\in\mathcal{P}} \theta(\phi_{ij}\eta_{ij}) & \text{if } \eta_{1n} = 0 \\ q^{n-2}\theta(\phi_{1n}\eta_{1n}) & \text{if } \eta_{1n} \neq 0, x_{\phi} \in Z(H_n) \\ 0 & \text{otherwise.} \end{cases}$$

2. The full group of upper-triangular matrices, $U_n(\mathbb{F}_q)$.
   Let $J = R^{+}$. Then

$$\overline{\chi^{\eta}(x_{\phi})} = \begin{cases} \displaystyle\prod_{(i,j)\in\operatorname{supp}(\eta)} q^{|\mathcal{P}_{i+1,l-1}| - |\operatorname{supp}(\phi)\bigcap\mathcal{P}_{i+1,l-1}|}\theta(\eta_{il}\phi_{il}) & \text{if } \begin{array}{c} \phi_{ij} \neq 0,\ \eta_{il} \neq 0 \Rightarrow j \geq l, \\ \phi_{jk} \neq 0,\ \eta_{ik} \neq 0 \Rightarrow j \geq i \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

General Results

For $\eta \in J^*$, let

$$\mathrm{ann}_J^R(\eta) = \{\rho \in J^* \mid X_\phi X_\rho \in \ker(\lambda_\eta),\ \text{for all } \phi \in J^*\}$$
$$\mathrm{ann}_J^L(\eta) = \{\rho \in J^* \mid X_\rho X_\phi \in \ker(\lambda_\eta),\ \text{for all } \phi \in J^*\}$$

Just like with normal character theories, there are also irreducible supercharacter theories. Persi Diaconis and I. Martin Isaacs give the following characterization of which supercharacters are irreducible characters.

Theorem 2. *Let J be closed, and $\eta \in J^*$. Then the supercharacher, χ^η, is* ***irreducible*** *if and only if $ann_J^R(\eta) \bigcup ann_J^L(\eta) = J^*$.*

This theorem implies several interesting combinatorial proposition, however, I will not be covering them here.

Theorem 3. *Let $H \subset U_n(\mathbb{F}_q)$ be an algebra group and suppose $char(\mathbb{F}_q) = p$. Then all nonzero supercharacter values are integer multiples of the pth roots of unity.*

A couple observations:

1. If $p = 2$, then all supercharacters are real-valued.

2. If $H = U_J$ is a pattern group, then a natural basis of V_H is $\{v^{(ij)} \mid (i,j) \in J\}$, where $v^{(ij)} \in \mathbb{F}_q^{|J|}$ is given by $v_{rs}^{(ij)} = \delta_{ij,rs}$. With this basis, the exact value of this multiple root of unity can be determined. Specifically,

$$\overline{\chi^\eta(x_\phi)} = \begin{cases} q^{\mathrm{corank}(\eta)-\mathrm{rank}(M_\theta^\eta)}\theta(b_0 * b_\phi^\eta) \displaystyle\prod_{(i,j)\in \mathrm{supp}(\phi)\bigcap \mathrm{supp}(\eta)} \theta(\phi_{ij}\eta_{ij}) & \text{if } \phi \text{ meshes with } \eta \\ 0 & \text{otherwise} \end{cases}$$

   where

$$(M_\phi^\eta)_{(i,j)(k,l)} \begin{cases} \phi_{jk}\eta_{il} & \text{if } (i,j,k,l) \in \mathcal{P} \\ 0 & \text{otherwise} \end{cases},$$

   $(b_\phi^\eta)_{(j,k)} = \displaystyle\sum_{(i,j,k)\in\mathcal{P}} \phi_{ij}\eta_{ik}$, and $b_0 \in \mathbb{F}_q^{|J|}$ satisfies $M_\phi^\eta b_0 = -(\displaystyle\sum_{(i,j,k)\in\mathcal{P}} \phi_{jk}\eta_{ik})$.

3. The above theorem is a supercharacter version of the Isaacs–Navarro conjecture, a generalization of McKay's conjecture.

Supercharacter Table Example

Consider the following group of order 16.

$$H = \begin{pmatrix} 1 & a & b & d \\ 0 & 1 & a & c \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ where } a, b, c, d \in \mathbb{F}_2.$$

Then this group has the presentation $H =< x, r, s \mid x^4 = r^2 = z^2 = 1, [x, r] = z, [x, z] = 1, [r, z] = 1 >$ where

$$x = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Notice that

$$l = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = x^2 r.$$

The character table for H is

	$\{e\}$	$\{z\}$	$\{lr\}$	$\{lrz\}$	$\{x, xz\}$	$\{xlr, xlrz\}$	$\{r, rz\}$	$\{l, lz\}$	$\{xr, xrz\}$	$\{xl, xlz\}$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	i	$-i$	1	-1	i	$-i$
χ_3	1	1	1	1	-1	-1	1	1	-1	-1
χ_4	1	1	-1	-1	$-i$	i	1	-1	$-i$	i
χ_5	1	1	1	1	1	1	-1	-1	-1	-1
χ_6	1	1	-1	-1	i	$-i$	-1	1	$-i$	i
χ_7	1	1	1	1	-1	-1	-1	-1	1	1
χ_8	1	1	-1	-1	$-i$	i	-1	1	i	$-i$
χ_9	2	-2	2	-2	0	0	0	0	0	0
χ_{10}	2	-2	-2	2	0	0	0	0	0	0

We can "compress" the character table together in the following way

$$\chi_1 = \chi_1, \quad \chi_x = \chi_3, \quad \chi_{lr} = \chi_5, \quad \chi_{rxl} = \chi_7$$

$$\chi_r = \chi_6 + \chi_8, \quad \chi_l = \chi_2 + \chi_4, \quad \chi_z = \chi_9 + \chi_{10}$$

to get a supercharacter table of H.

	$\{e\}$	$\{z\}$	$\{lr, lrz\}$	$\{x, xz, xlr, xlrz\}$	$\{r, rz\}$	$\{l, lz\}$	$\{xr, xrz, xl, xlz\}$
χ_1	1	1	1	1	1	1	1
χ_x	1	1	1	-1	1	1	-1
χ_{lr}	1	1	1	1	-1	-1	-1
χ_{rxl}	1	1	1	-1	-1	-1	1
χ_r	2	2	-2	0	-2	2	0
χ_l	2	2	-2	0	2	-2	0
χ_z	4	-4	0	0	0	0	0

However, this is not the only supercharacter theory and we can find many more. For example, let $\tilde{\chi}_{lr} = \chi_{lr} + \chi_{rxl}$. Then we also have the supercharacter theory

	$\{e\}$	$\{z\}$	$\{lr, lrz\}$	$\{x, xz, xlr, xlrz, xr, xrz, xl, xlz\}$	$\{r, rz\}$	$\{l, lz\}$
χ_1	1	1	1	1	1	1
χ_x	1	1	1	-1	1	1
$\tilde{\chi}_{lr}$	2	2	2	0	-2	-2
χ_r	2	2	-2	0	-2	2
χ_l	2	2	-2	0	2	-2
χ_z	4	-4	0	0	0	0

Once again, we can let $\tilde{\chi}_l = \chi_l + \chi_r + \tilde{\chi}_{lr}$ to get

	$\{e\}$	$\{z\}$	$\{l, r, lr, lz, rz, lrz\}$	$\{x, xz, xlr, xlrz, xr, xrz, xl, xlz\}$
χ_1	1	1	1	1
χ_x	1	1	1	-1
$\tilde{\chi}_l$	6	6	-2	0
χ_z	4	-4	0	0

And finally, let $\tilde{\chi}_x = \chi_l + \chi_r + \tilde{\chi}_{lr}$ to get

	$\{e\}$	$\{z\}$	$\{l, r, lr, lz, rz, lrz, x, xz, xlr, xlrz, xr, xrz, xl, xlz\}$
χ_1	1	1	1
$\tilde{\chi}_x$	7	7	-1
χ_z	4	-4	0

We cannot "compress" this supercharacter table any further.

Supercharacter Table Counterexample

Consider the character table for the group S_3. This group has no nontrivial supercharacter theories.

	$\{e\}$	(ab)	(abc)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$\longrightarrow$

	$\{e\}$	(ab, abc)
χ_1	1	1
$\chi_{2,3}$	3	-1 or 0

A interesting question for supercharacter theories in general is how many are there for any given finite group? Is there an explicit dependence on the set of irreducible characters? These are still unsolved questions.

$U_n(\mathbb{F})$ Supercharacter Table

Let $n = 4$ and $t = q - 1$. Then we have

$U_n(\mathbb{F})$	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	t	-1	t	t	-1	-1	t	-1	t	t	t	-1	t	t	t
χ_3	t	t	-1	t	-1	t	-1	-1	t	t	t	t	t	t	-1
χ_4	t	t	t	-1	t	-1	-1	-1	t	t	t	-1	t	t	t
χ_5	t^2	$-t$	$-t$	t^2	1	$-t$	$-t$	1	t^2	t^2	$-t$	t^2	t^2	t^2	$-t$
χ_6	t^2	$-t$	t^2	$-t$	$-t$	1	$-t$	1	t^2	t^2	$-t$	$-t$	t^2	t^2	t^2
χ_7	t^2	t^2	$-t$	$-t$	$-t$	$-t$	1	1	t^2	t^2	t^2	$-t$	t^2	t^2	$-t$
χ_8	t^3	$-t^2$	$-t^2$	$-t^2$	t	t	t	-1	t^3	t^3	$-t^2$	$-t^2$	t^3	t^3	$-t^2$
χ_9	tq	0	0	tq	0	0	0	0	$-q$	tq	0	$-q$	$-q$	tq	0
χ_{10}	tq	tq	0	0	0	0	0	0	tq	$-q$	$-q$	0	$-q$	tq	0
χ_{11}	t^2q	$-tq$	0	0	0	0	0	0	t^2q	$-tq$	q	0	$-tq$	t^2q	0
χ_{12}	t^2q	0	0	$-tq$	0	0	0	0	$-tq$	t^2q	0	q	$-tq$	t^2q	0
χ_{13}	t^2q^2	0	0	0	0	0	0	0	$-tq^2$	$-tq^2$	0	0	q^2	t^2q^2	0
χ_{14}	tq^2	0	tq	0	0	0	0	0	0	0	0	0	0	$-q^2$	$-q$
χ_{15}	t^2q^2	0	$-tq$	0	0	0	0	0	0	0	0	0	0	$-tq^2$	q