# TLDR

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# Topology.

## Part A

- A fundamental system of neighborhoods is a collection of open sets of $X$ s.t. $\forall x \in X$, $\exists$ open sets of $B_x$ s.t. $\forall b_x \in B_x$ where $b_x \ni x$ we have the following.
  $U \in \mathcal{T} \iff \forall x \in U \ \exists\ b_x \in \mathcal{B}_x$ s.t. $x \in b_x \subset U$
- $(X, \mathcal{T})$ is first countable if $\forall x \in X$, $\exists$ a countable neighborhood basis.
- $(X, \mathcal{T})$ is second countable if it admits a countable basis.
- $A = \mathrm{int}(A) \iff A$ is open $\qquad A = \overline{A} \iff A$ is closed
- Let $G$ be an infinite group. $G$ is residually finite (RF) if $\forall \gamma \in G \setminus \{e_G\}$, $\exists$ a finite group $Q$ & an epimorphism $\alpha: G \to Q$ s.t. $\alpha(\gamma) \neq e_G$.
  - Equivalently property: $\exists\ N \triangleleft G$ of finite index s.t. $\gamma \in N$
  - $G$ is RF iff it's Hausdorff with the profinite topology
- Let $f: X \to Y$ be a function. TFAE
  ① $f$ is continuous ② $\forall A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$ ③ $\forall$ closed $B \subset Y$, $f^{-1}(B) \subset X$ is closed ④ $\forall x \in X$ & open $V \subset Y$ s.t. $f(x) \in Y$, $\exists$ open $U \ni x$ s.t. $f(U) \subset V$.
- Let $f: X \to Y$ be continuous & injective & consider $f: X \to f(x) \subset Y$ endowed with the subspace topology. If this is a homeomorphism then $f$ is a topological embedding.
- Pasting Lemma: Let $X = A \cup B$ with $A, B \subset X$ closed. Let $f: A \to Y$, $g: B \to Y$ be continuous s.t. $\forall x \in A \cap B, f(x) = g(x)$. Then we can define a continuous function $h: X \to Y$ s.t. $h|_A = f$, $h|_B = g$.
- A collection of subsets $S_\alpha \subset X$ is called locally finite if $\forall x \in X$, $\exists\ U \ni x$ s.t. $U \cap S_\alpha \neq \emptyset$ for finitely many $\alpha$'s.
  - Qual Question: If locally finite $\Rightarrow$ have open property of compliment.
- Facts: ① Closed in compact is compact
  ② Compact in Hausdorff is closed
  ③ Image of compact is compact
  ④ Continuous bijection from compact to Hausdorff is homeomorphic
- Tube Lemma: Let $X \times Y$ be a product space with $Y$ compact. Let $N \subset X \times Y$ be an open set containing $\{x_0\} \times Y$. Then $\exists\ W \ni \{x_0\}$ open in $X$ s.t. $N \supset W \times Y$.
- $X$ is locally compact iff $\exists\ Y$ s.t. ① $X \subseteq Y$ ② $|Y \setminus X| = 1$ (typically call this point $\infty$) ③ $Y$ is compact & Hausdorff
  - $Y$ is unique up to a homeomorphism that restricts $f|_X: X \to X$.
- Continuous maps $f, f': X \to Y$ are homotopic if $\exists F: X \times I_t \to Y$ continuous s.t. $F(x,0) = f(x)$ and $F(x,1) = f'(x)$ $\forall x \in X$
  - Homotopy classes of paths creates a groupoid
  - Homotopy classes of loops based at a point creates a group, $\pi_1(X)$, the fundamental group.
    - Loops are paths $\Rightarrow$ collection of $\{\pi_1(x)\}_{x \in X}$ is a groupoid (get isomorphisms via $\hat{\alpha}: \pi_1(X, x_0) \to \pi_1(X, x_1)$ $\quad \hat{\alpha}[f] = [\bar{\alpha}][f][\alpha]$)
- Let $h: (X, x_0) \to (Y, y_0)$. $h_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is $h_*[f] = [h \circ f]$
- $P: E \to B$ Continuous surjection. $U \subset B$ is evenly covered if $p^{-1}(U) = \bigsqcup_\alpha V_\alpha$, for open $V_\alpha$, each of which is homeomorphic to $U$.
- $P: E \to B$ is a covering map if $\forall b \in B$, $\exists\ U \ni b$ open that's evenly covered
  - Continuous, surjective, locally homeo, & open maps. Not injective
- Lebesgue Number Lemma: Let $\mathcal{A}$ be a covering of a metric space $(X, d)$. If $X$ is compact, $\exists\ \delta > 0$ s.t. $\forall\ U \subset X$ w/ $\mathrm{diam}(U) < \delta$, $\exists\ \alpha \in \mathcal{A}$ s.t. $U \subset \alpha$.
- Lifting Lemma: Let $P: E \to B$ be a cover, $e_0 \in E$ with $p(e_0) = b_0$. If $F: I_s \to B$ is a path starting at $b_0$, $\exists!$ lift $\tilde{F}: I_s \to E$ beginning at $e_0$.
  - Homotopy Lifting Lemma: ↑ but $f: I_s \times I_t \to B$ $\Rightarrow \tilde{f}$ is a homotopy of paths
    - Can lift fundamental groups, but a lift of a loop is only guaranteed to be a path.
      - Can cover $S^1$ ($\pi_1(S^1) = \mathbb{Z}$) by $\mathbb{R}$ ($\pi_1(\mathbb{R}) = e$)
- General Lifting Lemma: $P: E \to B$ covering, $p(e_0) = b_0$, $f: Y \to B$ is continuous, $f(y_0) = b_0$. $Y$ path connected & locally path connected. Then $\exists!$ lift $\tilde{f}: Y \to E$ s.t. $\tilde{f}(y_0) = e_0$ iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0))$.
- Let $A \subset X$, $r: X \to A$ continuous is a retract if $r|_A = A$.
  - Deformation Retract if can homotopically get from $\mathrm{id}_X$ to the retract.
- Seifer-Van Kampen: $X = A \cup B$ open. $\pi_1(X) \cong \frac{\pi_1(A) * \pi_1(B)}{N}$ (N = take $\gamma$ in $A \cap B$ & compute $[\gamma]_A [\gamma^{-1}]_B$)

# Topology.

## Part B

- $E \sim E'$ if $\exists\, \varphi: E \xrightarrow{\sim} E'$ s.t. $\begin{array}{ccc} E & \xrightarrow{\varphi} & E' \\ & \searrow \circlearrowleft \swarrow & \\ & B & \end{array}$
- General Lifting Lemma: $p: E \to B$ cover w/ $p(e_0) = b_0$, $Y$ path connected & locally path connected, $f: Y \to B$ continuous w/ $f(y_0) = b_0$. Then $\exists\, \tilde{f}: Y \to E$ $(p \circ \tilde{f} = f)$ iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0))$. This lift is unique.
- A cover $p: E \to B$ is <u>universal</u> if $\pi_1(E) = \{\mathbb{1}\}$. Unique up to equivalence of covers.
- Let $p: E \to B$ be a cover. An equivalence $h: E \xrightarrow{\sim} E$ with $p \circ h = p$ is called a <u>covering/deck transformation.</u> $\mathcal{C}(E, p, B)$ makes a group.
  - $H_0 = p_*(\pi(E, e_0))$
  - $H_0 \triangleleft \pi_1(B, b_0)$ iff $\forall e_1, e_2 \in p^{-1}(b_0)$ $\exists$ covering transformation, $h$, s.t. $h(e_1) = e_2$.
- $p: E \to B$ regular iff $p_*(\pi_1(E)) \triangleleft \pi_1(B)$
- $p: X \to X/G$ quotient map iff $G \curvearrowright X$ properly discontinuously.
- Covering space of a graph is a graph
  - Universal covers are trees
- Every path in a graph is homotopic to a reduced edge path.
- $\pi_1(\bigvee_{i=1}^{n} S^1) \cong \mathbb{F}_n$ • If $X$ is a graph then $\pi_1(X)$ is free
- An $n$-simplex is the convex hull of $(n+1)$-points in $\mathbb{R}^{n+1}$ in general position (any 3 don't lie on a line)
- Boundary map: $\partial_n: C_n(X) \longrightarrow C_{n-1}(X)$ given by $\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$
  - $H_n(X) := \dfrac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}}$
- Simplicial Homology: $X$ has $\Delta$-complex structure. $C_n(X) :=$ free abelian group generated by $\sigma_\alpha: \Delta^n \to X$ for varying $\alpha$ but fixed $n$.
  - Triangulate
  - Orient edges based on vertex labels
  - Compute Boundary maps $\Rightarrow$ Compute Homology
- Singular Homology: Singular $n$-simplex is a continuous map $\sigma: \Delta^n \to X$. ("Singular" $\Rightarrow$ may not be injective)
  - Same idea as Simplicial
- Chain Complex is a collection of abelian groups $\{C_n\}_{n \geq 0}$ with group homomorphisms $\partial_n: C_n \to C_{n-1}$
- Relative Homology: $C_n(X, A) := C_n(X)/C_n(A)$
  - L.E.S.: $\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \to \cdots$ where $i$ is the inclusion, $j$ is surjective, $\delta: H_n(X, A) \to H_{n-1}(A)$ $[\alpha] \mapsto [\partial\alpha]$
  - $(X, A)$ is a <u>good pair</u> if $A$ is closed & $\exists\, V$ open, $A \subset V \subset X$, that deformation retracts to $A$.
    - Then $j: X \to X/A$ induces $H_n(X, A) \xrightarrow{\sim} H_n(X/A, A/A)$
    - In reduced homology $\tilde{H}_n(X, A) \cong H_n(X, A)$ (not good pair $\Rightarrow \tilde{H}_n(X, A) \cong 0$)
- Excision: If $Z \subset A \subset X$ s.t. $\bar{Z} \subset \operatorname{int}(A)$, then the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X \setminus Z, A \setminus Z) \xrightarrow{\sim} H_n(X, A)$
  - Equivalently: for $A, B \subset X$ with $X = \operatorname{int}(A) \cup \operatorname{int}(B)$, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \xrightarrow{\sim} H_n(X, A)$
- Given $\sigma = [v_0, \dots, v_n]$ $n$-simplex with $\sigma = \{\sum_{i=0}^{n} t_i v_i \mid {t_i \geq 0 \ \forall i \atop \sum t_i = 1}\}$, the <u>baricenter</u> of $\sigma$ is $b = \sum_{i=1}^{n} \frac{1}{n+1} v_i$
- <u>Baricentric Subdivision</u> of $\sigma$ is defined as follows:
  1. Subdivide every face of $\sigma$ & all of these faces by induction.
  2. For each $n{+}1$-simplex $[w_0, \dots, w_n]$ of the subdivision of the face of $\sigma$, $[b, w_0, \dots, w_{n-1}]$ is a simplex of the subdivision of $\sigma$.
  - Subdivision Operator: $S: C_n(X) \to C_n(X)$ $\sigma \mapsto \Sigma\, \sigma|_\tau$ ← non-empty $n$-simplex in the bsd of $\Delta^n$
- $f: S^n \to S^n$. $H_n(S^n) \cong \mathbb{Z} \Rightarrow f_*: H_n(S^n) \to H_n(S^n)$ $1 \mapsto d$. <u>$\deg(f) := d$</u>
  - (a) $\deg(id) = 1$ (b) $f$ not surjective $\Rightarrow \deg(f) = 0$ (c) $f \simeq g \Rightarrow \deg(f) = \deg(g)$ (d) $\deg(f \circ g) = \deg(f)\deg(g)$
  - (e) Reflection through hyperplane through $\vec{0} \Rightarrow \deg(f) = -1$ (f) If $f(x) = -x$ (antipodal), $\deg(f) = (-1)^{n+1}$ (g) $f$ has no fixed points $\Rightarrow \deg(f) = (-1)^{n+1}$
- Cellular Homology: $\partial_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta}\, e_\beta^{n-1}$ $d_{\alpha\beta} := \deg(\underset{\partial e_\alpha^n}{\overset{\shortparallel}{S_\alpha^{n-1}}} \to X^{n-1} \to S_\beta^{n-1})$
  - Ex: $X =$  $\cdots 0 \to \underset{\alpha}{\mathbb{Z}} \xrightarrow{\partial_2} \underset{a}{\mathbb{Z}} \oplus \underset{b}{\mathbb{Z}} \xrightarrow{\partial_1} \underset{x}{\mathbb{Z}} \oplus \underset{y}{\mathbb{Z}} \xrightarrow{\partial_0} 0$ $\partial_1(a) = x - y$ $\partial_1(b) = y - x$ $d_{\alpha a}$: boundary of $\alpha \to$ (  dotted part collapsed)  $= 1$ (goes around once) $d_{\alpha b} = 1$
    $\partial_2(e_2^\alpha) = a + b$ $\operatorname{Ker}(\partial_1) = \langle a + b \rangle \Rightarrow \operatorname{Im}(\partial_2) = \langle a + b \rangle \Rightarrow \operatorname{Ker}(\partial_2) = 0$ $\operatorname{Im}(\partial_1) = \langle x - y \rangle$ $H_0 \cong \mathbb{Z}$, $H_1 \cong 0$, $H_2 \cong 0$
- The <u>Euler characteristic</u> of a CW complex $X$ is $\chi(X) = \sum_n (-1)^n \cdot \#$ of $n$-cells
- Mayer–Viatoris: $X = \operatorname{int}(A) \cup \operatorname{int}(B)$. $\exists$ L.E.S. $\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots$
  $x \mapsto (x, -x)$ $x \oplus y \mapsto x + y$

# Topology.

## Part C

- The partial derivative operators on $\{\frac{\partial}{\partial x_1}|_p, \dots, \frac{\partial}{\partial x_k}|_p\}$ forms a basis for $T_p\mathcal{G}_p$.
- A differentiable map $i: N \hookrightarrow M$ is an immersion iff $\forall x \in N$ $T_{ix}: T_xN \to T_{i(x)}M$ is injective.
- An immersion is an embedding if it's also a topological embedding.
- $N^n \subset M^{n+k}$ is a $n$-dimensional submanifold iff $\forall p \in N$ $\exists$ a chart $(U, \varphi)$ around $p$, $\varphi: U \to U' \subset \mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$, s.t. $\varphi(N \cap U) = U' \cap (\mathbb{R}^n \times \{0\} \times \cdots \times \{0\})$
- A vector bundle is a triple $(E, \pi, X)$ where
  ① $\pi: E \to X$ is continuous ② $\forall x \in X$, $E_x: \pi^{-1}(x)$ is a $k$-dim vector space. ③ $\forall x \in X$, $\exists$ nbhd $U$ & homeo $f: \pi^{-1}(U) \to U \times \mathbb{R}^k$ s.t.
  $(f, U) =:$ Bundle chart $E =:$ Total Space $E_x =:$ Fiber
  $$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "f"] \arrow[dr, "\pi"'] & U \times \mathbb{R}^k \arrow[d, "proj_1"] \\ & U \end{tikzcd}$$
  (local triviality)
  - $E' \subset E$ is a subbundle provided $\forall x \in X$, $\exists$ bundle chart $(f, U)$ with $f(\pi^{-1}(U) \cap E') = U \times \mathbb{R}^k \times \{0\} \times \cdots \times \{0\}$
  - Given 2 vector bundles $E, E'$ over $X$, a continuous map $f: E \to E'$ is a bundle homomorphism if $\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "\pi"'] & E' \arrow[d, "\pi'"] \\ & X \end{tikzcd}$ & $f|_{E_x}: E_x \to E'_x$ is linear.
    - Rank Thm: Let $f: E \to F$ be a bundle homomorphism w/ constant rank $k$ $(rk(f|_{E_x}) = k)$.
      $\forall x \in X$, $\exists$ bundle charts $(\varphi, U)$ for $E$ & $(\psi, U)$ for $F$ s.t. $\psi \circ f \circ \varphi^{-1}|_{x \times \mathbb{R}^m}(v_1, \dots, v_m) = (v_1, \dots, v_k) = (v_1, \dots, v_k, 0, \dots, 0)$.
- A section is a continuous map $\sigma: X \to E$ s.t. $\pi \circ \sigma = id_X$
  - All sections are embeddings.
- A vector field on a smooth manifold is a smooth section of $TM$. $X: M \to TM$ $m \mapsto X(m) \in T_mM$
- Let $B$ & $\tilde{B}$ be ordered basis of an $n$-dim vector space, $V$. They have the same orientation if the linear map $L: B \mapsto \tilde{B}$ has determinant $> 0$.
  - An orientation of $E$ is a family of orientations on fibers that are locally constant. (think cylinder vs möbius band)
  - $M$ simply connected $\Rightarrow$ orientable
- A Riemannian metric $g$ on a smooth vector bundle $(E, \pi, M)$ is a choice of smoothly varying inner products on fibers of $E$.
  - A Riemannian Manifold is a smooth manifold w/ a Euclidean metric on $TM$.
- A family $\{\tau_\alpha\}_{\alpha \in A}$ of smooth functions, $\tau_\alpha: M \to [0,1]$, is called a partition of unity if $\forall x \in M$, $\exists$ nbhd $U_x \ni x$ s.t. $\tau_\alpha|_{U_x} \equiv 0$, except for finitely many $\alpha$'s & $\sum_{\alpha \in A} \tau_\alpha \equiv 1$. (a system of weighted averages)
- A partition of unity $\{\tau_\alpha\}_{\alpha \in A}$ is subordinate to cover $\mathcal{U}$ provided $\forall \alpha \in A$ $\exists$ $U_\alpha \in \mathcal{U}$ s.t. $Supp\, \tau_\alpha \subset U_\alpha$.
  - Every open cover of every smooth manifold has a subordinate partition of unity.
- Inverse Function Theorem: Let $F: M \to N$ be smooth, $p \in M$, & $T_pF: T_pM \xrightarrow{\sim} T_pN$ be an iso. Then $\exists$ connected nbhds $U_0 \ni p$ & $V_0 \ni f(p)$ s.t. $F|_{U_0}: U_0 \to V_0$ is diffeo.
- A smooth map is a submersion iff all of its differentials are onto (ie. $T_pF$ is onto $\forall p \in M$)
- Given $\Phi: M \to N$ smooth, $x \in M$ is a regular point iff $T_x\Phi$ is onto. Else $x$ is a critical point.
  - $V \in N$ is a regular value of $\Phi$ iff $\Phi^{-1}(v)$ consists only of regular points. Else it's a critical value.
- Sard's Theorem: If $\Phi: M \to N$ is smooth then almost every value is regular.
  - If $x_0 \in M$ is regular for $\Phi: M \to N$, $\exists$ nbhd $U \ni x_0$ st $\Phi|_U$ is regular.
- Regular Value Theorem: If $\Phi: M \to N$ is smooth & $v \in \Phi(M) \subset N$ is a regular value, then $\Phi^{-1}(v)$ is an embedded submanifold of $M$ with codimension $= \dim(N)$.
- Let $F: M^m \to N^n$ be a submersion. Then
  ⓐ $F$ is an open ⓑ Every point in $M$ is in the image of a smooth locally defined section of $F$. ⓒ $F$ surjective $\Rightarrow$ It's a quotient map.
- Let $F: M \to N$ be smooth & $S \subset N$ be a submanifold. $F$ is traverse to $S$ iff $\forall x \in F^{-1}(S)$, $span\{T_{F(x)}S, TF_x(T_xM)\} = T_xN$


  - Let $F: M \to N$ be smooth. If $F$ is transverse to $S$ then $F^{-1}(S)$ is a submanifold of $M$ whose codimension is equal to codim(S)
    Moreover, $\nu(F^{-1}(S)) \cong F^*(\nu(S))$ pullback
- Whitney Embedding Theorem: Every smooth $n$-manifold can be embedded into $\mathbb{R}^{2n+1}$ & immersed into $\mathbb{R}^{2n}$
- A family of subsets $\{C_\alpha\}_{\alpha \in A}$ of $X$ is locally finite iff $\forall x \in X$, $\exists$ nbhd $W_x$ s.t. $W_x \cap C_\alpha \neq \emptyset$ for finitely many $C_\alpha$'s.
- Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ be open covers on $X$. $\mathcal{V}$ is a refinement of $\mathcal{U}$ iff $\forall$ $V_\beta \in \mathcal{V}$, $\exists$ $U_\alpha \in \mathcal{U}$ s.t. $V_\beta \subset$
- A topological space is paracompact if every open cover has a locally finite refinement.
- Every open cover of every smooth manifold has a subordinate partion of unity
- Euclidean bump functions: $\exists$ $C^\infty$ functions $\lambda_n: \mathbb{R}^n \to \mathbb{R}$ s.t. $\lambda_n = \begin{cases} 1 & \text{on } B(0,1) \\ 0 & \text{on } \mathbb{R}^n \setminus B(0,2) \end{cases}$
  - If $W_i \subset V_i$ is of a "good cover" $\exists$ a $C^\infty$ function $\lambda_i: M \to [0,1]$ $\lambda_i|_{W_i} \equiv 1$ & $supp(\lambda_i) \subset V_i$


- Whitney Embedding Theorem (compact case): Let $M^n$ be a compact $n$-manifold. Then $\exists$ embedding $M \hookrightarrow \mathbb{R}^{2n+1}$ & immersion $M \hookrightarrow \mathbb{R}^{2n}$
- A smooth $\mathbb{R}$-action on $M$ is called a flow. For any curve $C_x: t \mapsto \underset{\text{grp action}}{\theta}(t, x)$ is called a flow line of $\theta$ through $x$.
- A velocity field is a vector field $Z: x \mapsto c_x'(0)$ that generates the flow $\theta$.
- A Lie Group is simultaneously a group and a smooth manifold.
  - The Lie group action $x \mapsto gx$ is a diffeomorphism.
  - Action is effective if $\ker(h) = e$ ("faithful") - For $p \in M$, an isotropy is $G_p = \{g \in G \mid gp = p\}$ ("Stablizer")
- A Lie bracket of $X$ & $Y$ in $\mathfrak{X}(M)$, is the map $[X, Y]: C^\infty M \to C^\infty M$ defined by $D_{[X,Y]}(f) = D_X D_Y f - D_Y D_X f$
  - Properties: ⓐ Bilinear: $[aV + bW, X] = a[V, X] + b[W, X]$ ⓑ Anti-symmetric: $[V, W] = -[W, V]$ ⓒ Jacobi Identity: $[V, [W, X]] + [X, [V, W]] + [W, [X, V]] = 0$ ⓓ $[fV, gW] = fg[V, W] + (fD_V g)W - (gD_W f)V$
- A Lie algebra is a vector space $\mathfrak{g}$ w/ multiplication $[,]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying ⓐ Bilinearity ⓑ Anti-symmetry ⓒ Jacobi Identity